UNIT - I

BETA AND GAMMA FUNCTIONS

Gamma Function: [In Mathematics, the Gamma Function (Represented by the capital Greek Letterγ) is an extension of the factorial function, with its argument shifted down by 1, to real and complex number]

Def: The definite integral $\int_0^\infty e^{-x}\, x^{n-1} dx$ is called the Gamma function and is denoted by $\Gamma(n)$ and read as "Gamma n"

The integral converges only for n>0

Thus, $\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1} dx$ where n>0

Gamma function is also called Eulerian integral of the second kind.

Note: The integral $\int_0^\infty e^{-x}\, x^{n-1} dx$ does not converge if n≤0

Properties of Gamma Function:

1. To show that $\Gamma(1) = 1$

Sol. By the def of Gamma function; we have

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx$$

$$\therefore\ \Gamma(1) = \int_0^\infty e^{-x}\, x^{1-1} dx = \int_0^\infty e^{-x}\, x^0 dx = \int_0^\infty e^{-x}\, dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty$$

$$= -[e^{-\infty} - e^0] = -[0-1] = 1$$

$$\therefore\ \Gamma(1) = 1$$

2. To show that $\Gamma(n) = (n-1)\Gamma(n-1)$ where n>1.

Sol. By the def of Gamma function; we have

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1} dx$$

$$= \left[x^{n-1}\frac{e^{-x}}{(-1)}\right]_0^\infty - \int_0^\infty (n-1)x^{n-2}\left(\frac{e^{-x}}{-1}\right) dx \text{ (Integrate by parts)}$$

$$= \underset{x\to\infty}{lt}\ \frac{x^{n-1}}{e^x} + 0 + (n-1)\int_0^\infty e^{-x}\, x^{n-2} dx$$

$$=(n-1)\int_0^\infty e^{-x} x^{n-2} dx \qquad \left(\because\ \underset{n\to\infty}{lt}\ \frac{x^{n-1}}{e^x} = 0 \text{ for } n>1\right)$$

$= (n-1)\,\Gamma(n-1)$

$\therefore\ \Gamma(n) = (n-1)\,\Gamma(n-1)$

Note: 1. $\Gamma(n+1) = n\Gamma(n)$

2. If n is a +ve fraction then we can write.

$\Gamma(n) = (n-1)(n-2)............(n-r)\Gamma(n-r)$ Where (n-r)>0

3. If n is a non-negative integer, then $\Gamma(n+1) = n!$

Proof: From property II, We have.

$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)$ (by property II again)

$= n(n-1)(n-2)\,\Gamma(n-2)$ (by property II again)

$= n(n-1)(n-2)(n-3)\,\Gamma(n-3)$

$= n(n-1)(n-2)(n-3)\ \ldots..3.2.1\ \Gamma(1)$

$= n(n-1)(n-2)(n-3)\ \ldots.3.2.1 \qquad \because\ \Gamma(1) = 1$

= n!

$\therefore\ \Gamma(n+1) = n!\ (n = 0,1,2 \ldots \ldots..)$

This shows that the Gamma function can be regarded as a generalization of the elementary factorial function.

1. Solve $\Gamma\left(\frac{9}{2}\right)$

Sol. $\Gamma\left(\frac{9}{2}\right) = \left(\frac{9}{2}-1\right)\Gamma\left(\frac{9}{2}-1\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{7}{2}\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)$

$= \frac{7}{2}.\frac{5}{2}.\Gamma\left(\frac{5}{2}\right) = \frac{7}{2}.\frac{5}{2}.\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)$

$= \frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\Gamma\left(\frac{3}{2}\right) = \frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)$

$= \frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)$

2. Solve $\Gamma\left(13/3\right)$

Sol: $\Gamma\left(13/3\right) = \frac{10}{3}.\frac{7}{3}.\frac{4}{3}.\frac{1}{3}.\Gamma\left(1/3\right)$

Note: When n is a –ve fraction

We have $\Gamma(n+1) = n\Gamma(n)$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

3. Compute $\Gamma\left(-1/2\right)$

Sol. We have $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

Put $n = \left(\frac{-1}{2}\right)$

$$\Gamma\left(-1/2\right) = \frac{\Gamma\left(1/2\right)}{\frac{-1}{2}} = -2\sqrt{\pi}$$

4. Compute $\Gamma\left(-5/2\right)$

Sol. We have $\Gamma(n) = \frac{\Gamma(n+1)}{n}$ `12

$$\Gamma\left(-5/2\right) = \frac{\Gamma\left(-3/2\right)}{\frac{-5}{2}} = \frac{-2}{5}\Gamma\left(-3/2\right)$$

$$= \frac{-2}{5}.\frac{\Gamma\left(\frac{-3}{2}+1\right)}{\frac{-3}{2}} = \frac{2^2}{5.3}\Gamma\left(-1/2\right)$$

$$= \frac{2^2}{15}.\frac{\Gamma\left(-1/2+1\right)}{\frac{-1}{2}} = \frac{-2^3}{15}\Gamma\left(1/2\right) = \frac{-2^3}{15}\sqrt{\pi} \quad = \frac{-8}{15}.\sqrt{\pi}$$

Beta Function:

Def: The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called the Beta function and is denoted by $\beta(m,n)$ and read as "Beta m, n". The above integral converges for m>0, n>0

Thus, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, where m>0, n>0

Beta function is also called Eulerian integral of the first kind.

Properties of Beta Function:

(i). **Symmetry of Beta function**; i.e., $\beta(m,n) = \beta(n,m)$

Proof: By the def, we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$put\ (1-x) = y\ so\ that\ dx = -dy$$

When $x=1 \Rightarrow y=0$

$x=0 \Rightarrow y=1$

$$\therefore \beta(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1}(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1} dy$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx \qquad \because \int_a^b f(t)dt = \int_a^b f(x)dx$$

$$= \beta(n,m)$$

$$\therefore \beta(m,n) = \beta(n,m)$$

Aliter : We know that $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

From properties of definite integrals, we have

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\therefore \beta(m,n) = \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1} dx$$

$$= \int_0^1 (1-x)^{m-1}x^{n-1}dx$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx = \beta(n,m)$$

$$\therefore \beta(m,n) = \beta(n,m)$$

(ii). Prove that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$

Proof: By the def, we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Put $x = \sin^2\theta$ *so that* $dx = 2\sin\theta\cos\theta\, d\theta$

$$\Rightarrow dx = \sin 2\theta\, d\theta$$

When $x = 1 \Rightarrow \theta = \pi/2$ and $x = 0 \Rightarrow \theta = 0$

$$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1} 2\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta \sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\ d\theta$$

$$\therefore \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\ d\theta$$

Note: $\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta = \frac{1}{2}\beta(m,n)$

(iii) Prove that $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

Proof: By the def, we have

$$\beta(m+1,n) + \beta(m,n+1) = \int_0^1 x^m(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^n dx$$

$$= \int_0^1 \left[x^m(1-x)^{n-1} + x^{m-1}(1-x)^n\right]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}[x + (1-x)]dx$$

$$=\int_0^1 x^{m-1}(1-x)^{n-1}dx=\beta(m,n)$$

Hence $\beta(m,n)=\beta(m+1,n)+\beta(m,n+1)$

(iv). If m and n are positive integers, then $\beta(m,n)=\dfrac{(m-1)!(n-1)!}{(m+n-1)!}$

Proof: By the def, we have

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx \text{ ....(1)}$$

$$=\left[x^{m-1}\frac{(1-x)^n}{n(-1)}\right]_0^1-\int_0^1\frac{(1-x)^n}{n(-1)}(m-1)x^{m-2}dx \qquad \text{(Integration by parts)}$$

$$=\frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^n dx=\frac{m-1}{n}\beta(m-1,n+1) \text{ .... (2)}$$

Now we have to find $\beta(m-1,n+1)$

To obtain this put m=m-1 and n=n+1 in equation. (1), we have

$$\beta(m-1,n+1)=\frac{m-2}{n+1}\beta(m-2,n+2)$$

From Equation. (2)

$$\beta(m,n)=\frac{m-1}{n}.\frac{m-2}{n+1}\beta(m-2,n+2) \text{ .... (3)}$$

Changing m to m-2 and n to n+2, from (1) we have

$$\beta(m-2,n+2)=\frac{m-3}{n+2}\beta(m-3,n+3)$$

From Equation (3), we have

$$\beta(m,n)=\frac{m-1}{n}.\frac{m-2}{n+1}.\frac{m-3}{n+2}\beta(m-3,n+3) \text{..................... (4)}$$

Proceeding like this, we get

$$\beta(m,n)=\frac{(m-1)(m-2)(m-3)......[m-(m-1)]}{n(n+1)(n+2).....(n+m-2)}\beta[m-(m-1),n+(m-1)]$$

$$=\frac{(m-1)(m-2)(m-3).....1}{n(n+1)(n+2).....(n+m-2)}\beta(1,n+m-1) \text{..................... (5)}$$

But $\beta(1, n+m-1) = \int_0^1 x^0 (1-x)^{n+m-2} dx = \int_0^1 (1-x)^{n+m-2} dx$

$$= \left[\frac{(1-x)^{n+m-1}}{(n+m-1)(-1)} \right]_0^1 = \frac{-1}{n+m-1}(0-1) = \frac{1}{n+m-1}$$

From equation (5), we have

$$\beta(m,n) = \frac{(m-1)(m-2)(m-3).......1}{n(n+1)(n+2)....(n+m-2)(n+m-1)} = \frac{(m-1)!}{n(n+1)(n+2).......(n+m-2)(n+m-1)}$$

Multiplying the numerator and denominator by (n-1)!, we have

$$\beta(m,n) = \frac{(m-1)!(n-1)!}{(n+m-1)(n+m-2)....(n+2)(n+1)n(n-1)!} = \frac{(m-1)!(n-1)!}{(n+m-1)!}$$

$$\therefore \beta(m,n) = \frac{(m-1)!(n-1)!}{(n+m-1)!}$$

Note 1: Putting m=1 in $\beta(m,n) = \frac{(m-1)!(n-1)!}{(n+m-1)!}$, we have

$$\beta(1,n) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

2: By putting n=1, we get $\beta(m,1) = \frac{1}{m}$

Other forms of Beta Function:

1. To Show

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx (or) \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy (or) \beta(p,q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy$$

Proof: By the def, we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1+x)^{n-1} dx \;......................(1)$$

Put $x = \frac{1}{1+y}$ so that $dx = \frac{-dy}{(1+y)^2}$

when $x = 0 \Rightarrow y = \infty$ *and* $x = 1 \Rightarrow y = 0$

From equation (1), we have

$$\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1} \left(1 - \frac{1}{1+y}\right)^{n-1} . \frac{-dy}{(1+y)^2}$$

$$=\int_0^\infty \frac{1}{(1+y)^{m-1}}\cdot\frac{y^{n-1}}{(1+y)^{n-1}}\cdot\frac{dy}{(1+y)^2}$$

$$=\int_0^\infty \frac{y^{n-1}}{(1+y)^{m-1+n+1+2}}\,\mathrm{dy}=\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,\mathrm{dy}$$

$$\therefore \beta(m,n)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,\mathrm{dx}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Again since Beta function is symmetrical in m and n, we also have

$$\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,\mathrm{dx}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ (3)$$

Hence $\beta(m,n)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,\mathrm{dx}=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,\mathrm{dx}$

2. **To show** $\beta(m,n)=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,\mathrm{dx}$

Proof: We have

$$\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx=\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx+\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Now consider $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

Put $x=\frac{1}{y}$ so that $dx=-\frac{1}{y^2}dy,$

When $x=1 \Rightarrow y=1$ *and* $x\to\infty \Rightarrow y=0$

$$\therefore \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,\mathrm{dx}=\int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{(1+\frac{1}{y})^{m+n}}\left(-\frac{1}{y^2}\right)\mathrm{dy}$$

$$=\int_0^1 \frac{\frac{1}{y^{m-1}}}{\frac{(1+y)^{m+n}}{y^{m+n}}}\cdot\frac{1}{y^2}\,\mathrm{dx}=\int_0^1 \frac{1}{y^{m-1}}\,\frac{y^{m+n}}{(1+y)^{m+n}}\,\frac{1}{y^2}\,\mathrm{dy}$$

$$=\int_0^1 \frac{y^{m+n-m+1-2}}{(1+y)^{m+n}}dy=\int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}dy=\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Hence Equation (1) becomes

$$\beta(\mathrm{m},\mathrm{n})=\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,\mathrm{dx}+\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,\mathrm{dx}$$

$$\therefore \beta(\mathrm{m},\mathrm{n})=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,\mathrm{dx}$$

3. **To show** $\beta(\mathrm{m,n}) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(\mathrm{ax+b})^{m+n}} \mathrm{dx}$

Proof: We have, $a^m b^n \int_0^\infty \frac{x^{m-1}}{(\mathrm{ax+b})^{m+n}} \mathrm{dx} = a^m b^n \int_0^\infty \frac{x^{m-1}}{b^{m+n}\left(\frac{a}{b}\mathrm{x}+1\right)^{m+n}} \mathrm{dx}$

Put $\frac{ax}{b} = y. Then\, dx = \frac{b}{a}.dy\, and\; x = \frac{by}{a}$

When $x = 0 \Rightarrow y = 0$ *and* $x \to \infty \quad \Rightarrow y \to \infty$

$$\therefore a^m b^n \int_0^\infty \frac{x^{m-1}}{(\mathrm{ax+b})^{m+n}} \mathrm{dx} = \frac{a^m b^n}{b^{m+n}} \int_0^\infty \frac{\left(\frac{by}{a}\right)^{m-1}}{(1+\mathrm{y})^{m+n}} \frac{b}{a} \mathrm{dy}$$

$$= a^m b^{n-m-n} \int_0^\infty \frac{b^{m-1}}{a^{m-1}} \frac{y^{m-1}}{(1+\mathrm{y})^{m+n}} \frac{b}{a} \mathrm{dy}$$

$$= a^m b^{-m} \frac{b^m}{a^m} \int_0^\infty \frac{y^{m-1}}{(1+\mathrm{y})^{m+n}} \mathrm{dy}$$

$$\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \beta(m,n)$$

4. **To show** $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(\mathrm{x}+a)^{m+n}} dx = \frac{\beta(m,n)}{a^n(1+a)^m}$

Proof: By the def, we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Put $x = \frac{(1+a)y}{y+a}$

$$dx = (1+a)\left[\frac{(y+a)1 - y(1+0)}{(y+a)^2}\right] dy = \frac{a(1+a)}{(y+a)^2} dy$$

When $x = 0 \quad \Rightarrow y = 0$ *and* $x = 1 \quad \Rightarrow y = 1$

Now equation (1) becomes

$$\beta(m,n) = \int_0^1 \frac{(1+a)^{m-1} y^{m-1}}{(y+a)^{m-1}} . \left(1 - \frac{(1+a)y}{y+a}\right)^{n-1} \frac{a(1+a)}{(y+a)^2} dy$$

$$= \int_0^1 \frac{(1+a)^{m-1} y^{m-1}}{(y+a)^{m-1}} . \left(\frac{y+a-y-ay}{y+a}\right)^{n-1} \frac{a(1+a)}{(y+a)^2} dy$$

$$= \int_0^1 \frac{a(1+a)^m y^{m-1}}{(y+a)^{m-1+n-1+2}} . (a-ay)^{n-1} dy$$

$$= \int_0^1 \frac{a(1+a)^m y^{m-1}}{(y+a)^{m+n}} . a^{n-1}(1-y)^{n-1} dy$$

$$= a^n(1+a)^m \int_0^1 \frac{y^{m-1}(1-y)^{n-1}}{(y+a)^{m+n}}dy = a^n(1+a)^m \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}dx$$

$$\therefore \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}dx = \frac{\beta(m,n)}{a^n(1+a)^m}$$

5. To show $\int_b^a (x-b)^{m-1}(\mathrm{a}-x)^{n-1}dx = (a-b)^{m+n-1}\beta(m,n), m>0, n>0.$

Proof: We have

$$\beta(\mathrm{m,n}) = \int_0^1 x^{m-1}(1-x)^{n-1}$$

Put $x = \frac{y-b}{a-b}$ $so\,that\, dx = \frac{dy}{a-b}$

When $x=0 \Rightarrow y=b$ and $x=1 \Rightarrow y=a$

$$\therefore \beta(\mathrm{m,n}) = \int_b^a \left(\frac{y-b}{a-b}\right)^{m-1}\left[1-\left(\frac{y-b}{a-b}\right)\right]^{n-1}\frac{dy}{a-b}$$

$$= \int_b^a \frac{(y-b)^{m-1}}{(a-b)^{m-1}}.\frac{(a-b-y+b)^{n-1}}{(a-b)^{n-1}}\frac{dy}{a-b}$$

$$= \int_b^a \frac{(y-b)^{m-1}(a-\mathrm{y})^{n-1}}{(a-b)^{m-1+n-1+1}}\mathrm{dy} = \int_b^a \frac{(x-b)^{m-1}(a-x)^{n-1}}{(a-b)^{m+n-1}}dx$$

$$\therefore \int_b^a \frac{(\mathrm{x}-b)^{m-1}(a-x)^{n-1}}{(a-b)^{m+n-1}}dx = (a-b)^{m+n-1}\beta(m,n)$$

PROBLEMS

1. S.T $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$

Sol: $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta d\theta = \int_0^{\frac{\pi}{2}} \sin^{m-1}\theta\cos^{n-1}\theta(\sin\theta\cos\theta)d\theta$

$$= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{\frac{m-1}{2}}(\cos^2\theta)^{\frac{n-1}{2}}(\sin\theta\cos\theta)d\theta$$

Put $\sin^2\theta = x\, so\, that \sin\theta\cos\theta\, d\theta = \frac{dx}{2}$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta d\theta = \frac{1}{2}\int_0^1 x^{\frac{m-1}{2}}(1-x)^{\frac{n-1}{2}}dx$$

$$= \frac{1}{2}\int_0^1 x^{\left(\frac{m+1}{2}\right)-1}(1-x)^{\left(\frac{n+1}{2}\right)-1}dx = \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$$

Aliter: We have $\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta = \frac{1}{2}\beta(m,n)$........................ (1)

Put $p=2m-1, q=2n-1$ so that $m=\frac{p+1}{2}$ and $n=\frac{q+1}{2}$

Then (1) becomes $\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta d\theta=\frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$

or $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta d\theta=\frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$

2. Solve $\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$

Sol: Put $x^2=y$ so that $dx=\frac{dy}{2x}=\frac{1}{2}y^{\frac{-1}{2}}dy$

When $x=0\Rightarrow y=0, and\ x=1\Rightarrow y=1$

$$\therefore \int_0^1 \frac{x}{\sqrt{1-x^2}}dx=\int_0^1 \frac{y^{\frac{1}{2}}}{\sqrt{1-y}}\frac{1}{2}y^{\frac{-1}{2}}dy$$

$$=\frac{1}{2}\int_0^1 y^0(1-y)^{\frac{-1}{2}}dy=\frac{1}{2}\int_0^1 y^{1-1}(1-y)^{\frac{1}{2}-1}dy=\frac{1}{2}\beta\left(1,\frac{1}{2}\right)$$

3. Solve $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Sol: Put $x^2=9y$ so that dx=$\frac{3}{2}.y^{\frac{-1}{2}}dy$

$$\therefore \int_0^3 \frac{dx}{\sqrt{9-x^2}}=\int_0^3 \left(9-x^2\right)^{\frac{-1}{2}}dx=\int_0^1 (9-9y)^{\frac{-1}{2}}.\frac{3}{2}y^{\frac{-1}{2}}dy$$

$$=\frac{3}{2}\int_0^1 y^{\frac{-1}{2}}\frac{1}{3}(1-y)^{\frac{-1}{2}}dy$$

$$=\frac{1}{2}\int_0^1 y^{\frac{1}{2}-1}(1-y)^{\frac{1}{2}-1}dy=\frac{1}{2}\beta\left(\frac{1}{2},\frac{1}{2}\right)$$

4. S.T $\int_0^1 x^m\left(1-x^n\right)^p dx=\frac{1}{n}\beta\left(\frac{m+1}{n},p+1\right)$

Sol: Put $x^n=y$ so that $dx=\frac{1}{n}y^{\frac{1}{n}-1}dy$

$$\therefore \int_0^1 x^m\left(1-x^n\right)^p dx=\int_0^1 y^{m/n}(1-y)^p\frac{1}{n}y^{\frac{1-n}{n}}dy$$

$$=\frac{1}{n}\int_0^1 y^{\frac{m+1-n}{n}}(1-y)^p dy$$

$$=\frac{1}{n}\int_0^1 y^{\frac{m+1}{n}-1}(1-y)^{(p+1)-1}dy=\frac{1}{n}\beta\left(\frac{m+1}{n},p+1\right)$$

5. S.T $\int_{-1}^1 (1+x)^{m-1}(1-x)^{n-1}dx=2^{m+n-1}\beta(m,n)$

Sol: We have $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $x=\frac{1+y}{2}$ so that $dx=\frac{1}{2}dy$

$$\therefore \beta(m,n)=\int_{-1}^{1}\frac{(1+y)^{m-1}}{2^{m-1}}\left(1-\frac{1+y}{2}\right)^{n-1}.\frac{1}{2}dy$$

$$=\int_{-1}^{1}\frac{(1+y)^{m-1}(1-y)^{n-1}}{2^{m+n-1}}dy=\frac{1}{2^{m+n-1}}\int_{-1}^{1}(1+x)^{m-1}(1-x)^{n-1}dx$$

$$\therefore \int_{-1}^{1}(1+x)^{m-1}(1-x)^{n-1}dx=2^{m+n-1}\beta(m,n)$$

6. P.T $\int_0^1 \frac{xdx}{\sqrt{1-x^5}}=\frac{1}{5}\beta\left(\frac{2}{5},\frac{1}{2}\right)$

Sol: Put $x^5=y\Rightarrow x=y^{\frac{1}{5}}$ so that $dx=\frac{1}{5}y^{\frac{1}{5}-1}dy=\frac{1}{5}.y^{\frac{-4}{5}}dy$

When $x=0\Rightarrow y=0,$ and $x=1\Rightarrow y=1$

$$\therefore \int_0^1\frac{xdx}{\sqrt{1-x^5}}=\int_0^1\frac{y^{\frac{1}{5}}}{\sqrt{1-y}}.\frac{1}{5}\,y^{\frac{-4}{5}}dy=\frac{1}{5}\int_0^1 y^{\frac{-3}{5}}(1-y)^{\frac{-1}{2}}dy$$

$$=\frac{1}{5}\int_0^1 y^{\frac{2}{5}-1}(1-y)^{\frac{1}{2}-1}dy=\frac{1}{5}\beta\left(\frac{2}{5},\frac{1}{5}\right)$$

7. Evaluate $\int_0^1\frac{x^2dx}{\sqrt{1-x^5}}$ **in terms of Beta function**

Sol: Put $x^5=y\Rightarrow x=y^{\frac{1}{5}}$ so that $dx=\frac{1}{5}y^{\frac{-4}{5}}dy$

When $x=0\Rightarrow y=0,$ and $x=1\Rightarrow y=1$

$$\therefore \int_0^1\frac{x^2dx}{\sqrt{1-x^5}}=\int_0^1\frac{y^{\frac{2}{5}}}{(1-y)^{\frac{1}{2}}}\frac{1}{5}\,y^{\frac{-4}{5}}dy=\frac{1}{5}\int_0^1 y^{\frac{-2}{5}}\,(1-y)^{\frac{-1}{2}}dy$$

$$=\frac{1}{5}\int_0^1 y^{\frac{3}{5}-1}(1-y)^{\frac{1}{2}-1}dy=\frac{1}{5}\beta\left(\frac{3}{5},\frac{1}{2}\right)$$

8. S.T $\int_a^b (x-a)^m(b-x)^n\,dx=(b-a)^{m+n+1}\beta(m+1,n+1)$

Sol: Put $x=a+(b-a)y\,(or)\left[x=\frac{y-a}{b-a}\right]$ so that $dx=(b-a)dy$

When $x=a\Rightarrow y=0,$ and $x=b\Rightarrow y=1$

$$\therefore \int_a^b (x-a)^m(b-x)^n\,dx=\int_0^1[(b-a)\,y]^m\,[b-a-(b-a)y]^n\,(b-a)dy$$

$$=\int_0^1(b-a)^m\,y^m(b-a)^n(1-y)^n(b-a)dy$$

$$= (\mathrm{b}-a)^{m+n+1}\int_0^1 y^m (1-y)^n\,dy$$

$$= (\mathrm{b}-a)^{m+n+1}\int_0^1 y^{(m+1)-1}(1-y)^{(n+1)-1}\,dy$$

$$= (b-a)^{m+n+1}\beta(m+1,n+1)$$

9. S.T $\displaystyle\int_0^\infty \frac{x^{m-1}}{(x+a)^{m+n}}dx = a^{-n}\beta(m,n)$

Sol: We have $\displaystyle\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

Put $x=\dfrac{y}{a}$ so that $dx=\dfrac{dy}{a}$

$$\therefore \beta(m,n)=\int_0^\infty \frac{y^{m-1}}{a^{m-1}(1+\frac{y}{a})^{m+n}}\frac{dy}{a}=\frac{1}{a^m}\int_0^\infty \frac{y^{m-1}.\mathrm{a}^{m+n}}{(a+y)^{m+n}}dy$$

$$= a^n\int_0^\infty \frac{y^{m-1}}{(y+a)^{m+n}}dy$$

$$\therefore \frac{1}{a^n}\beta(m,n)=\int_0^\infty \frac{x^{n-1}}{(\mathrm{x}+a)^{m+n}}dx$$

Hence $\displaystyle\int_0^\infty \frac{x^{m-1}}{(x+a)^{m+n}}dx = a^{-n}\beta(m,n)$

Relation between $\beta-\gamma$ function

P.T. $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, m>0, n>0

Proof: By the def of $\Gamma-$function

$$\Gamma(m)=\int_0^\infty e^{-x}\,x^{m-1}dx$$

Put $x=t^2 \Rightarrow dx = 2t\,dt$

When $x=0 \Rightarrow t=0$ and $x=\infty \Rightarrow t=\infty$

$$\therefore \Gamma(m)=\int_0^\infty e^{-t^2}(t^2)^{m-1}2tdt=\int_0^\infty e^{-t^2}t^{2m-2+1}2dt$$

$$\Gamma(m)=2\int_0^\infty e^{-t^2}t^{2m-1}dt$$

$$\therefore \Gamma(m)=2\int_0^\infty e^{-x^2}x^{2m-1}dx$$

Similarly,

$$\Gamma(n)=2\int_0^\infty e^{-y^2}y^{2n-1}dy$$

$$\therefore \Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1} dx.\int_0^\infty e^{-y^2} y^{2n-1} dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dxdy$$

Transforming to polar coordinates

$x = r\cos\theta, y = r\sin\theta$ and $dxdy = rdrd\theta$

r is varies from 0 to ∞ and θ is varies from 0 to $\pi/2$

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2} r^{2m-1}\cos^{2m-1}\theta r^{2n-1}\sin^{2n-1}\theta rdrd\theta$$

$$= 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2} r^{2(m+n)-1}\cos^{2m-1}\theta\sin^{2n-1}\theta drd\theta$$

$$= 2\int_0^\infty e^{-r^2} r^{2(m+n)-1} dr.2\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta$$

$$= \Gamma(m+n).\beta(m,n)$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

1. S.T $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Sol: We know that $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, m>0, n>0

Taking m=n=$\frac{1}{2}$, we have $\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2$ ....(1)$\left[\because \Gamma(1) = 1\right]$

But $\beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1} dx = \int_0^1 x^{\frac{-1}{2}}(1-x)^{\frac{-1}{2}} dx$

Put $x = \sin^2\theta$ so that dx=2sinθcos$\theta d\theta$

When $x = 0 \Rightarrow \theta = 0, and\ x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta}\frac{1}{\cos\theta}.2\sin\theta\cos\theta d\theta = 2\int_0^{\frac{\pi}{2}} d\theta = 2[\theta]_0^{\frac{\pi}{2}}$$

$$= 2\left[\frac{\pi}{2} - 0\right] = \pi$$

From Equation (1) $\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

2. To show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Sol: We have $\Gamma(n)=\int_0^\infty e^{-x^2}x^{n-1}dx$

Taking $n=\frac{1}{2}$, we have $\Gamma\left(\frac{1}{2}\right)=\int_0^\infty e^{-x}\,x^{\frac{-1}{2}}dx$

Put $x=t^2\ so\,that\,dx=2tdt$

When $x=0\Rightarrow t=0\ and\ x=\infty\Rightarrow t=\infty$

$$\therefore\Gamma\left(\frac{1}{2}\right)=\int_0^\infty e^{-t^2}(t^2)^{\frac{-1}{2}}2tdt=2\int_0^\infty e^{-t^2}dt$$

$$(or)2\int_0^\infty e^{-x^2}dx=\Gamma\left(\frac{1}{2}\right)\qquad\left(\because\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right)$$

$$\Rightarrow\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$$

Note: **1.** $\int_{-\infty}^0 e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$

2. $\int_{-\infty}^\infty e^{-x^2}dx=\sqrt{\pi}$

3. $\Gamma(n)$ is defined when n>0

4. $\Gamma(n)$ is defined when 'n' is a negative fraction.

5. But $\Gamma(n)$ is not defined when n=0 and 'n' is a negative integer

3. P.T. $\Gamma(n)\Gamma(n-1)=\frac{\pi}{\sin n\pi}$

Proof: We know that $\beta(m,n)=\int_0^\infty\frac{x^{n-1}}{(1+x)^{m+n}}dx$

Also we have $\beta(m,n)=\frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$

$$\therefore\int_0^\infty\frac{x^{n-1}}{(1+x)^{m+n}}dx=\frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$$

Taking m+n=1 so that m=1-n, we get

$$\int_0^\infty\frac{x^{n-1}}{1+x}dx=\frac{\Gamma(1-n)\Gamma(m)}{\Gamma(1)}\qquad\because\Gamma(1)=1$$

(or) $\Gamma(n)\Gamma(1-n)=\int_0^\infty\frac{x^{n-1}}{1+x}dx$...........(1)

We have $,\int_0^\infty\frac{x^{2m}}{1+x^{2n}}dx=\frac{\pi}{2n}\operatorname{cosec}\frac{(2m+1)\pi}{2n}$ Where m>0, n>0 and n>m

Put $x^{2n} = t$ and $\frac{2m+1}{2n} = s$, we have

$$\int_0^\infty \frac{t^{2m/2n}.t^{1/2n}}{(2n)(1+t)t}dt = \frac{\pi}{2n}\operatorname{cosec} s\pi$$

$$(or)\ \frac{1}{2n}\int_0^\infty \frac{t^{2m/2n}.t^{1/2n-1}}{1+t}dt = \frac{\pi}{2n}\operatorname{cosec} s\pi$$

$$(or)\ \int_0^\infty \frac{t^{\frac{2m+1}{2n}-1}}{1+t}dt = \pi\operatorname{cosec} s\pi$$

$$(or)\int_0^\infty \frac{t^{s-1}}{1+t}dt = \frac{\pi}{\sin s\pi}$$

$$(or)\int_0^\infty \frac{x^{s-1}}{1+x}dx = \frac{\pi}{\sin s\pi}$$

$$(or)\int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}.........(2)$$

From equation (1) and (2) we have $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

4. S.T. $\Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx, n > 0$

Sol: We have $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$ …………………..(1)

Putting $x = \log\frac{1}{y} = -\log y$

$(or)\ y = e^{-x}$ so that $dy = -e^{-x}dx$

$dx = \frac{-1}{y}dy$

Equation (1) becomes

$$\Gamma(n) = -\int_1^0 \left(\log \frac{1}{y}\right)^{n-1}.y.\frac{1}{y}dy = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$$

$$\therefore \Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$$

5. Evaluate i. $\int_0^1 x^4(1-x)^2 dx$ **ii.** $\int_0^2 x\left(8-x^3\right)^{\frac{1}{3}} dx$

Sol. (i). $\int_0^1 x^4(1-x)^2 dx = \int_0^1 x^{5-1}(1-x)^{3-1} dx = \beta(5,3)$

$$= \frac{\Gamma(5)\Gamma(3)}{\Gamma(5+3)} = \frac{\Gamma(5).\Gamma(3)}{\Gamma(8)} = \frac{4!2!}{7!} = \frac{4!2}{7\times6\times5\times4!} = \frac{1}{105}$$

(ii). $Let\ x^3 = 8y \Rightarrow x = 2y^{\frac{1}{3}} \Rightarrow dx = \frac{2}{3}.y^{\frac{-2}{3}}dy$

When $x=0 \Rightarrow y=0\, and\, x=2 \Rightarrow y=1$

$$\therefore \int_0^2 x\left(8-x^3\right)^{\frac{1}{3}} dx = \int_0^1 2y^{\frac{1}{3}}(8-8y)^{\frac{1}{3}}.\frac{2}{3}y^{\frac{-2}{3}}dy$$

$$=\frac{8}{3}\int_0^1 y^{\frac{-1}{3}}(1-y)^{\frac{-1}{3}}dy = \frac{8}{3}\int_0^1 y^{2/3-1}(1-y)^{4/3-1}dy$$

$$=\frac{8}{3}\beta\left(\frac{2}{3},\frac{4}{3}\right) \qquad \left[\because \Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}\right]$$

$$=\frac{8}{9}\frac{\pi}{\sin\left(\pi/3\right)} = \frac{16\pi}{9\sqrt{3}}$$

6. Evaluate $\int_0^{\frac{\pi}{2}} \sin^5\theta\cos^{\frac{7}{2}}\theta d\theta$

Sol. We have $\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m,n)$........(1)

Put $2m-1=5 \Rightarrow m=3$ and $2n-1=\frac{7}{2} \Rightarrow n=\frac{9}{4}$

$\therefore$ Equation (1) becomes

$$\int_0^{\frac{\pi}{2}} \sin^5\theta\cos^{\frac{7}{2}}\theta\, d\theta = \frac{1}{2}\beta\left(3,\frac{9}{4}\right) = \frac{1}{2}.\frac{\Gamma(3).\Gamma\left(9/4\right)}{\Gamma\left(3+9/4\right)}$$

$$=\frac{1}{2}.\frac{\Gamma(3)\Gamma\left(9/4\right)}{\Gamma\left(21/4\right)} = \frac{1}{2}.\frac{2!.\Gamma\left(9/4\right)}{\Gamma\left(21/4\right)}$$

$$=\frac{\Gamma\left(9/4\right)}{\frac{17}{4}.\frac{13}{4}.\frac{9}{4}\Gamma\left(9/4\right)} = \frac{64}{1989}$$

7. Evaluate (i). $\int_0^\infty 3^{-4x^2} dx$

Sol. Since $3=e^{\log 3}$

$$\therefore 3^{-4x^2} = e^{-4x^2\log 3}$$

$$\int_0^\infty 3^{-4x^2} dx = \int_0^\infty e^{-4x^2\log 3} dx$$

Put $2x\sqrt{\log 3} = y$ *so that* $dx = \frac{dy}{2\sqrt{\log 3}}$

$$\therefore \int_0^\infty 3^{-4x^2}dx = \int_0^\infty e^{-y^2}\frac{dy}{2\sqrt{\log 3}} = \frac{1}{2\sqrt{\log 3}}.\int_0^\infty e^{-y^2}dy$$

$$= \frac{1}{2\sqrt{\log 3}}.\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4\sqrt{\log 3}} = \sqrt{\frac{\pi}{16\log 3}}$$

8. When n is a +ve integer. P.T. $2^n\Gamma\left(n+\frac{1}{2}\right)=1.3.5....(2n-1)\sqrt{\pi}$

Sol. We know that $\Gamma(n+1)=n\Gamma(n)$…………………….. (1)

$$\therefore \Gamma\left(n+\frac{1}{2}\right)=\Gamma\left(n-\frac{1}{2}+1\right)=\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$$

$$=\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{3}{2}+1\right)=\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right).\Gamma\left(n-\frac{3}{2}\right)$$

$$=\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right).\left(n-\frac{5}{2}\right).\Gamma\left(n-\frac{5}{2}\right)$$

$$=\frac{2n-1}{2}.\frac{2n-3}{2}.\frac{2n-5}{2}.\Gamma\left(\frac{2n-5}{2}\right)$$

$$=\frac{2n-1}{2}.\frac{2n-3}{2}.\frac{2n-5}{2}.......\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)$$

$$=\frac{(2n-1)(2n-3)(2n-5)....3.1}{2^n}.\sqrt{\pi}$$

$$\therefore 2^n\Gamma\left(n+\frac{1}{2}\right)=(2n-1)(2n-3)(2n-5).....1\sqrt{\pi}$$

9. P.T. $2^{2n-1}\Gamma(n).\Gamma\left(n+\frac{1}{2}\right)=\Gamma(2n)\sqrt{\pi}$

Sol. By def, we have $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

$$(or)\int_0^1 x^{n-1}(1-x)^{m-1}dx=\beta(n,m)=\frac{\Gamma(n).\Gamma(m)}{\Gamma(n+m)}\text{...........(1)}$$

Put $x=\sin^2\theta\ so\,that\,dx=2\sin\theta\cos\theta\,d\theta$

From equation (1) $\int_0^{\frac{\pi}{2}}\sin^{2n-2}\cos^{2m-2}(2\sin\theta\cos\theta)d\theta=\frac{\Gamma(n).\Gamma(m)}{\Gamma(n+m)}$

$$(or)\int_0^{\frac{\pi}{2}}\sin^{2n-1}\theta\cos^{2m-1}\theta\,d\theta=\frac{\Gamma(n).\Gamma(m)}{2\Gamma(n+m)}\text{ ………………………….. (2)}$$

Putting $m=\frac{1}{2}\,in\,equation(2),we\,get$

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta d\theta = \frac{\Gamma(n).\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(n+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{2}.\frac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .. (3)$$

Now putting m=n in equation (2), we get

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{(\Gamma(n))^2}{2\Gamma(2n)}$$

$$(or)\frac{(\Gamma(n))^2}{2\Gamma(2n)} = \frac{1}{2^{2n-1}}\int_0^{\pi/2} (2\sin\theta\cos\theta)^{2n-1}\theta\, d\theta = \frac{1}{2^{2n-1}}\int_0^{\pi/2} \sin^{2n-1} 2\theta\, d\theta$$

$$= \frac{1}{2^{2n-1}}.\frac{1}{2}\int_0^{\pi/2} \sin^{2n-1}\varphi\, d\varphi (put\, 2\theta = \varphi) \quad = \frac{1}{2^{2n}} 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\varphi\, d\phi$$

$$= \frac{(\Gamma(n))^2}{2\Gamma(2n)} = \frac{1}{2^{2n-1}}.\frac{\sqrt{\pi}}{2}.\frac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)}$$

$$\Rightarrow 2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = \sqrt{\pi}.\Gamma(2n)$$

UNIT-II

Special Functions.

Legendre's D.E:-

The diff Eqn $(1-x^2)y'' - 2xy' + n(n+1)y = 0 \rightarrow (1)$ is known as the Legendre's diff Eqn. Here 'n' is a real number. But in most applications only integral values of n are needed.

The Legendre's Eqn (1) can also be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0.$$

General solution of Legendre's Eqn:-

The Legendre's Eqn (1) can be solved in series of ascending or descending powers of x. The solution in descending powers of x is more important than the one in ascending powers.

Let us assume that the solution of (1) in series is

$$y = \sum_{r=0}^{\infty} a_r x^{k-r}.$$

Then

$$y' = \sum_{r=0}^{\infty} a_r (k-r) x^{k-r-1}$$

$$y'' = \sum_{r=0}^{\infty} a_r (k-r)(k-r-1) x^{k-r-2}.$$

substituting the values of y, y' and y'' in (1), we have

$$(1-x^2)\sum_{r=0}^{\infty} a_r x^{k-r}(k-r-1)x^{k-r-2}(k-r) - 2x\sum_{r=0}^{\infty} a_r (k-r)x^{k-r-1}$$

$$+ n(n+1)\sum_{r=0}^{\infty} a_r x^{k-r} = 0.$$

(or) $$\sum_{r=0}^{\infty} a_r\left[(k-r)(k-r-1)x^{k-r-2} + \{n(n+1) - (k-r)(k-r-1) - 2(k-r)\}x^{k-r}\right] = 0.$$

(or) $\sum_{r=0}^{\infty} a_r \left[(K-r)(K-r-1) x^{K-r-2} + \{ n(n+1) - (K-r)(K-r+1) \} x^{K-r} \right] = 0.$

(or) $\sum_{r=0}^{\infty} a_r \left[(K-r)(K-r-1) x^{K-r-2} + \{ n^2 - (K-r)^2 + n - (K-r) \} x^{K-r} \right] = 0.$

(or) $\sum_{r=0}^{\infty} a_r \left[(K-r)(K-r-1) x^{K-r-2} + (n-K+r)(n+K-r+1) x^{K-r} \right] = 0.$ → ②

Equating the coefficients of highest powers of x,

i.e., of x^K to 0, we get

$a_0 (n-K)(n+K+1) = 0.$

putting $r = 0$ in coefficient of x^{K-r} in ②

But, $a_0 \neq 0$ because it is the coefficient of the first term of the series to start with.

Hence, either $K = n$ (or) $K = -(n+1)$ — ③

Now equating the coefficient of next lower power of x,

i.e., of x^{K-1} in ③ to zero, we get

$a_1 (n-K+1)(n+K) = 0.$

putting $r = 1$ in coefficient of x^{K-r} in ②,

$a_1 = 0$ because neither $(n-K+1)$ nor $(n+K)$ is zero by virtue of ③

Further equating the coefficient of the general term,

i.e., of x^{K-r} in ③ to zero, we get

$a_{r-2} (K-r+2)(K-r+1) + (n-K+r)(n+K-r+1) a_r = 0$

Putting $r = r-2$ in the coefficient of x^{k-r-2}

$$a_r = \frac{-(k-r+2)(k-r+1)}{(n-k+r)(n+k-r+1)} a_{r-2} \longrightarrow (4).$$

Putting $r = 3$ in (4), we get

$$a_3 = -\frac{(k-1)(k-2)}{(n-k+3)(n+k-2)} a_1 \quad [\because a_1 = 0]$$

$$= 0$$

Similarly, we can show by putting $r = 5, 7, 9, \ldots$ in (4),

that a_5, a_7, a_9, etc, are zero.

i.e., $a_{2k+1} = 0, \forall k \in N$.

Now, in order to evaluate $a_2, a_4, a_6, \ldots$ we consider two cases:

<u>case: I</u> <u>when $k = n$</u>

In this case from (4), we have

$$a_r = -\frac{(n-r+2)(n-r+1)}{r(2n-r+1)} a_{r-2}.$$

putting $r = 2, 4, \ldots$ etc., we get

$$a_2 = -\frac{n(n-1)}{2(2n-1)} a_0.$$

$$a_4 = -\frac{(n-2)(n-3)}{4(2n-3)} a_2$$

$$= -\frac{(n-2)(n-3)}{4(2n-3)} \left[-\frac{n(n-1)}{2(2n-1)}\right] a_0$$

$$= \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} a_0.$$

and similarly $a_6, a_8, \ldots$ etc. can be found.

$\therefore y = a_0 x^n + a_2 x^{n-2} + a_4 x^{n-4} + \cdots \quad [\because a_{2k+1} = 0, \ \forall k \in N]$

(or)

$$y = a_0 \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x^{n-4} \cdots \right] \rightarrow (6)$$

which is one solution of Legendre's equation.

<u>case II</u> <u>when $k = -(n+1)$</u>

From (4), we have $a_r = \dfrac{(n+r-1)(n+r)}{r(2n+r+1)} a_{r-2}$

putting $r = 2, 4, \cdots$ etc, we get

$$a_2 = \frac{(n+1)(n+2)}{2(2n+3)} a_0$$

$$a_4 = \frac{(n+3)(n+4)}{4(2n+5)} a_2$$

$$= \frac{(n+3)(n+4)}{4(2n+5)} \left[\frac{(n+1)(n+2)}{2(2n+3)} a_0 \right]$$

$$= \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 \cdot (2n+3)(2n+5)} a_0 .$$

and similarly $a_6, a_8, \cdots$ etc, can be found.

$$\therefore y = \sum_{r=0}^{\infty} a_r x^{-n-1-r}$$

$$= a_0 x^{-n-1} + a_2 x^{-n-3} + a_4 x^{-n-5} + \cdots$$

$$= a_0 \left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 \cdot (2n+3)(2n+5)} x^{-n-5} + \cdots \right] \rightarrow (7)$$

which is the other solution of Legendre's equation.

Legendre's function of first kind, $P_n(x)$

When n is a positive integer and $a_0 = \frac{1.3.5 \cdots (2n-1)}{n!}$, the solution ⑥ above is denoted by $P_n(x)$ and is called the Legendre's function of first kind.

Therefore,

$$P_n(x) = \frac{1.3.5 \cdots (2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} + \cdots\right] \rightarrow ⑧.$$

$P_n(x)$ is a terminating series and gives what are known as Legendre's polynomials for different values of n.

We can write $$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n r!(n-2r)!(n-r)!} x^{n-2r} \rightarrow ⑨$$

where $N = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd} \end{cases}$

Legendre's function of second kind $[Q_n(x)]$

When n is a positive integer and $a_0 = \frac{n!}{1.3.5 \cdots (2n+1)}$, the above solution ⑦ is called the Legendre's function denoted by $Q_n(x)$ and is called the Legendre's function of second kind

$$Q_n(x) = \frac{n!}{1.3.5 \cdots (2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2.(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}x^{-n-5} + \cdots\right] \rightarrow ⑩$$

But, $Q_n(x)$ is an infinite or non-terminating series, as n is positive and $Q_n(x)$ is therefore not a polynomial.

Thus, most general solution of Legendre's eqn is given by

$$\boxed{y = a P_n(x) + b Q_n(x)}$$ where a & b are arbitary constants.

Rodrigue's formula

* Statement: we shall prove that

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

proof: Let us assume $y = (x^2-1)^n$ — ①

Then $\frac{dy}{dx} = n(x^2-1)^{n-1}(2x)$

$\frac{dy}{dx} = 2xn(x^2-1)^{n-1}$

$y_1 = 2xn(x^2-1)^{n-1}$

$y_1 = 2xn\frac{(x^2-1)^n}{x^2-1}$

$\Rightarrow y_1(x^2-1) = 2xn(x^2-1)^n$

i.e., $(x^2-1)y_1 = 2xny$ [From ①]

diff $(n+1)$ times by Leibnitz's theorem, we have

$$(x^2-1)y_{n+2} + (n+1)2x \cdot y_{n+1} + \frac{n(n+1)}{2!} \cdot 2 \cdot y_n = 2n[xy_{n+1} + (n+1)y_n]$$

(or) $(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$

(or) $(1-x^2)y_{n+2} - 2xy_{n+1} + n(n+1)y_n = 0$

(or) $(1-x^2)\frac{d^2(y_n)}{dx^2} - 2x\frac{d(y_n)}{dx} + n(n+1)y_n = 0$

(or) $(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + n(n+1)z = 0$, where $z = y_n$

which is Legendre's eqn and its solution is

$z = c P_n(x)$ (or) $y_n = c P_n(x)$ —— ②

To determine constant c, put $x=1$.

Then $(y_n)_{x=1} = c P_n(1) = c$ $[\because P_n(1) = 1]$ or $c = (y_n)_{x=1}$ —③

From ①, we have $y = (x^2-1)^n = [(x+1)(x-1)]^n$

$= (x+1)^n (x-1)^n$

diff n times by Leibnitz's theorem, we have

$y_n = \left\{\frac{d^n}{dx^n}(x+1)^n\right\}(x-1)^n + {}^nc_1\left\{\frac{d^{n-1}}{dx^{n-1}}(x+1)^n\right\}$

$\frac{d}{dx}(x-1)^n + \cdots + {}^nc_1 \frac{d}{dx}(x+1)^n\left\{\frac{d^{n-1}}{dx^{n-1}}(x-1)^n\right\} + (x+1)^n$

$\left\{\frac{d^n}{dx^n}(x-1)^n\right\}.$

$y_n = \{n!\}(x-1)^n + n\left\{\frac{n!}{1!}(x+1)\right\}n(x-1)^{n-1} + \cdots + n.n(x+1)^{n-1}$

$\left\{\frac{n!}{1!}(x-1)\right\} + (x+1)^n\{n!\}.$

putting $x=1$ on both sides, we get

$(y_n)_{x=1} = (1+1)^n n!$

$= 2^n n!$

From ③, $c = (y_n)_{x=1} = 2^n n!$

From ②, we have $y_n = 2^n n! P_n(x)$

(or) $P_n(x) = \frac{1}{2^n n!} y_n = \frac{1}{2^n n!}\frac{d^n(y)}{dx^n} = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$

This is known as Rodrigue's formula.

<u>Legendre polynomials</u>

we know the Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n \text{ —— ①}$$

Putting $n=0$ in eqn ①, we get

$$P_0(x) = 1.$$

putting $n=1$ in Eqn ①, we get

$$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x.$$

putting $n=2$ in Eqn ①, we get

$$P_2(x) = \frac{1}{(4)(2!)}\frac{d^2}{dx^2}(x^2-1)^2$$

$$= \frac{1}{8}\frac{d}{dx}\left[2(x^2-1)(2x)\right]$$

$$= \frac{1}{2}\frac{d}{dx}(x^3-x)$$

$$= \frac{1}{2}(3x^2-1)$$

putting $n=3$ in Eqn ①, we get

$$P_3(x) = \frac{1}{8(3!)}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{8\times\cancel{6}}\frac{d^2}{dx^2}\left[\cancel{3}(x^2-1)^2(\cancel{2}x)\right]$$

$$= \frac{1}{8}\frac{d^2}{dx^2}\left[(x^2-1)^2(x)\right]$$

$$= \frac{1}{8}\frac{d}{dx}\left[x(2)(x^2-1)(2x) + (x^2-1)^2\right]$$

$$= \frac{1}{8}\frac{d}{dx}\left[4(x^4-x^2) + (x^2-1)^2\right]$$

$$= \frac{1}{8}\frac{d}{dx}\left[4x^4-4x^2+x^4+1-2x^2\right]$$

$$= \frac{1}{8}\frac{d}{dx}\left[5x^4-6x^2+1\right]$$

$$= \frac{1}{8}(20x^3-12x)$$

$$= \frac{1}{\cancel{8}_2}\left[\cancel{4}(5x^3-3x)\right]$$

Note :- These formulae can also be derived directly from the series expansion for $P_n(x)$.

Generating Function For $P_n(x)$

Theorem :- To show that $P_n(x)$ is the coefficient of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$

(or) To show that $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$

(or) prove that $\dfrac{1}{\sqrt{1-2xt+t^2}} = P_0(x)+P_1(x)t+P_2(x)t^2+ \cdots$

Proof :- LHS $(1-2xt+t^2)^{-1/2}$

Expanding $(1-2xt+t^2)^{-1/2}$ by Binomial theorem, we have

$$(1-2xt+t^2)^{-1/2} = [1-t(2x-t)]^{-1/2}$$

$$= 1+\frac{1}{2}t(2x-t)+\frac{1\cdot 3}{2\cdot 4}t^2(2x-t)^2$$

$$+ \cdots + \frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n-2)}t^{n-1}(2x-t)^{n-1} + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}t^n(2x-t)^n + \cdots$$

coefficient of t^n in this expansion

$$= \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}(2x)^n + \frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdots(2n-2)}\left[\text{coeff of } t \text{ in } (2x-t)^{n-1}\right]$$

$$+ \frac{1\cdot 3\cdots(2n-5)}{2\cdot 4\cdots(2n-4)}\left[\text{coeff of } t^2 \text{ in } (2x-t)^{n-2}\right] + \cdots$$

$$= \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}(2x)^n + \frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdots(2n-2)}\left[-(n-1)\cdot(2x)^{n-2}\right]$$

$$+ \frac{1\cdot 3\cdots(2n-5)}{2\cdot 4\cdots(2n-2)}\left[\frac{(n-2)(n-3)}{2!}(2x)^{n-4}\right] + \cdots$$

$$= \frac{1\cdot 3 \cdots (2n-1)}{2^n \cdot n!}(2^n x^n) - \frac{1\cdot 3 \cdots (2n-3)}{2\cdot 4 \cdots (2n-2)}\left[(n-1)2^{n-2}x^{n-2}\right]$$

$$+ \frac{1\cdot 3 \cdots (2n-5)}{2\cdot 4 \cdots (2n-2)}\left[\frac{(n-2)(n-3)}{2!}2^{n-4}x^{n-4}\right] \cdots$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot (2n-1)(2n-3)}x^{n-4} \cdots\right]$$

$= P_n(x)$, by defn.

$$\therefore [1-2xt+t^2]^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Since $P_n(x)$ has been generated by the function $(1-2xt+t^2)^{-1/2}$ we call this function as its generating function.

Orthogonality of Legendre polynomials

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n \end{cases}$$

(or) $$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1}\delta_{mn}$$

where δ_{mn} is called 'kronecker delta' and is defined by $\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$

Laplace's First and second integral of $P_n(x)$.

(a) **Laplace's First integral for $P_n(x)$:**

If n is a positive integer, then

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x \pm \sqrt{x^2-1}\cos\phi\right]^n d\phi.$$

(b) **Laplace's second integral for $P_n(x)$:**

If 'n' is a positive integer, then

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x \pm \sqrt{x^2-1}\cos\phi\right]^{-(n+1)} d\phi.$$

Orthogonality of Legendre Polynomials

we shall prove that $\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n \end{cases}$

(or) $\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1}\, \delta_{mn}$

where δ_{mn} is called 'Kronecker delta' and is defined by $\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$

Proof: case (i) $m \neq n$

we know that Legendre's Equation is

$$(1-x^2) y'' - 2xy' + n(n+1) y = 0$$

This Eqn can also be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1) y = 0$$

Now as $P_n(x)$ is a solution of it, so we have

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1) P_n = 0 \quad \text{--- (1)}$$

similarly, $P_m(x)$ is also a solution of it, so we have

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + m(m+1) P_m = 0 \quad \text{--- (2)}$$

multiplying (1) by $P_m(x)$ and (2) by $P_n(x)$,

$$P_m(x)\left[\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1) P_n\right] = 0 \rightarrow (3)$$

$$P_n(x)\left[\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + m(m+1) P_m\right] = 0 \rightarrow (4)$$

subsbract ③ & ④, we get

$$P_m(x)\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}-P_n(x)\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}$$

$$+\left[n(n+1)-m(m+1)\right]P_m(x)P_n(x)=0.$$

Integrating this eqn w.r.to x from -1 to 1, we have

$$\int_{-1}^{1}P_m\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx-\int_{-1}^{1}P_n(x)\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}dx$$

$$+\left[n(n+1)-m(m+1)\right]\int_{-1}^{1}P_m(x)P_n(x)\,dx=0.$$

$$\Rightarrow\left[P_m\left\{(1-x^2)\frac{dP_n}{dx}\right\}\right]_{-1}^{1}-\int_{-1}^{1}\frac{dP_m}{dx}\left[\left\{(1-x^2)\frac{dP_n}{dx}\right\}\right]dx$$

$$-\left[P_n\left\{(1-x^2)\frac{dP_m}{dx}\right\}\right]_{-1}^{1}+\int_{-1}^{1}\frac{dP_n}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right]dx$$

$$+\left[n(n+1)-m(m+1)\right]\int_{-1}^{1}P_m(x)P_n(x)\,dx=0$$ [∵ Integrating by parts]

$$\Rightarrow\left[n(n+1)-m(m+1)\right]\int_{-1}^{1}P_m(x)P_n(x)\,dx=0$$

$$\Rightarrow(n-m)(n+m+1)\int_{-1}^{1}P_m(x)P_n(x)\,dx=0$$

Since m and n are non-negative integers, $n+m+1\neq0$.

so if $n-m\neq0$, i.e., $n\neq m$, we get $\int_{-1}^{1}P_m(x)P_n(x)\,dx=0$ — ③

This is known as the orthogonality property of Legendre polynomials.

Case (ii): when $m=n$

we have from Rodrigue's formula,

$$(n!\,2^n)^2 \int_{-1}^{1} P_n^2(x)\,dx = \int_{-1}^{1} D^n (x^2-1)^n\, D^n (x^2-1)^n\, dx$$

[Integration by parts]

$$= \left\{ D^n (x^2-1)^n\, D^{n-1} (x^2-1)^n \right\}_{-1}^{1} - \int_{-1}^{1} D^{n+1} (x^2-1)^n\, D^{n-1} (x^2-1)^n\, dx$$

since $D^{n-1}(x^2-1)^n$ has x^2-1 as a factor, the 1st term on the right vanishes for $x = \pm 1$.

Thus,

$$(n!\,2^n)^2 \int_{-1}^{1} P_n^2(x)\,dx = -\int_{-1}^{1} D^{n+1} (x^2-1)^n\, D^{n-1} (x^2-1)^n\, dx$$

[Integrate by parts $(n-1)$ times]

$$= (-1)^n \int_{-1}^{1} D^{2n} (x^2-1)^n\, (x^2-1)^n\, dx$$

$$= (-1)^n \int_{-1}^{1} (2n)!\, (x^2-1)^n\, dx$$

$$= 2(2n)! \int_{0}^{1} (1-x^2)^n\, dx$$

put $x = \sin\theta$

$dx = \cos\theta\, d\theta$

limits ; upper limit : $x = 0 \Rightarrow 0 = \sin\theta$

$\Rightarrow \theta = 0$

lower limit : $x = 1 \Rightarrow 1 = \sin\theta$

$\Rightarrow \theta = \pi/2$

$$= 2(2n)! \int_{0}^{\pi/2} \cos^{2n+1}\theta\, d\theta$$

$$= 2(2n)! \; \frac{2n(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 3\cdot 1}$$

$$= 2(2n)! \frac{[2n(2n-2)\cdots 4\cdot 2]^2}{(2n+1)!}$$

$$= \frac{2}{(2n+1)!}\left(2^n n!\right)^2$$

(or) $$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$$

Thus $$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n. \end{cases}$$

Recurrence Relations (or) Formulae

1. $(2n+1)x\,P_n(x) = (n+1)P_{n+1}(x) + n\,P_{n-1}(x)$.

Proof: we know that $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$ — ①

diff both sides w.r.to 't', we have

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}\cdot(-2x+2t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x).$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-3/2} = 0+\sum_{n=1}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-1/2}(1-2xt+t^2)^{-1} = \sum_{n=1}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-1/2} = \sum_{n=1}^{\infty}(1-2xt+t^2)\,n t^{n-1} P_n(x)$$

$$= (1-2xt+t^2)\sum_{n=1}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (x-t)\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)\sum_{n=1}^{\infty} n t^{n-1} P_n(x)$$

[∵ from ①]

$$\Rightarrow (x-t)\left[P_0(x) + tP_1(x) + t^2P_2(x) + \cdots t^{n-1}P_{n-1}(x) + t^nP_n(x) + \cdots\right] = (1-2xt+t^2)\left[P_1(x) + 2tP_2(x) + 3t^2P_3(x) + \cdots + (n-1)t^{n-2}P_{n-1}(x) + n t^{n-1}P_n(x) + \cdots + (n+1)t^nP_{n+1}(x) + \cdots\right]$$

Now equating the coefficients of t^n on both sides of this eqn, we get

$$x P_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$(2n+1)x\,P_n(x) = (n+1)P_{n+1}(x) + n\,P_{n-1}(x),\ n \geq 1.$$

Note: Replacing n by $(n-1)$ in this result, we get

$$(2n-1)x\,P_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x)$$

$$nP_n(x) = (2n-1)x\,P_{n-1}(x) - (n-1)P_{n-2}(x),\ n \geq 2.$$

II. $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

(or) $xP_n'(x) = nP_n(x) + P_{n-1}'(x)$.

proof:- we know that $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$ — ①

diff this eqn w.r.to 't', we have

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} nt^{n-1}P_n(x)$$

$$(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} nt^{n-1}P_n(x) \rightarrow ②$$

Again, diff eqn ① w.r.to x, we have.

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2t) = \sum_{n=0}^{\infty} t^n P_n'(x)$$

$$t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} t^n P_n'(x) \rightarrow ③$$

dividing ③ by ② and cross multiplying, we get

$$t\sum_{n=0}^{\infty} nt^{n-1}P_n(x) = (x-t)\sum_{n=0}^{\infty} t^n P_n'(x)$$

$$\sum_{n=0}^{\infty} n t^{n-1} t P_n(x) = x\sum_{n=0}^{\infty} t^n P_n'(x) - t\sum_{n=0}^{\infty} t^n P_n'(x).$$

$$\sum_{n=0}^{\infty} n t^n P_n(x) = x\sum_{n=0}^{\infty} t^n P_n'(x) - \sum_{n=0}^{\infty} t^{n+1} P_n'(x).$$

$$\sum_{n=0}^{\infty} n t^n P_n(x) = x\sum_{n=0}^{\infty} t^n P_n'(x) - \sum_{n=1}^{\infty} t^n P_{n-1}'(x).$$

Equating the coefficients of t^n on both sides, we get

$$\boxed{nP_n(x) = xP_n'(x) - P_{n-1}'(x)}$$

3) $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$.

proof:- From recurrence relation I, we have

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x).$$

diff both sides w.r.to 'x', we have.

$$(2n+1)\left[x P_n'(x) + P_n(x)\right] = (n+1) P_{n+1}'(x) + n P_{n-1}'(x).$$

$$\Rightarrow (2n+1) x P_n'(x) + (2n+1) P_n(x) = (n+1) P_{n+1}'(x) + n P_{n-1}'(x) \quad \text{—} (1)'$$

From recurrence relation II, we have

$$x P_n'(x) = n P_n(x) + P_{n-1}'(x).$$

$\therefore$ From (1), we have

$$(2n+1)\left[n P_n(x) + P_{n-1}'(x)\right] + (2n+1) P_n(x) = (n+1) P_{n+1}'(x) + n P_{n-1}'(x)$$

$$(2n+1)\left[n P_n(x) + P_n(x)\right] = (n+1) P_{n+1}'(x) - (2n+1) P_{n-1}'(x) + n P_{n-1}'(x)$$

$$(2n+1)(n+1) P_n(x) = (n+1) P_{n+1}'(x) - (n+1) P_{n-1}'(x).$$

$$= (n+1)\left[P_{n+1}'(x) - P_{n-1}'(x)\right]$$

$$\boxed{(2n+1) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)}.$$

Note: Prove that $P_n' - P_{n-2}' = (2n-1) P_{n-1}$

Replace n by $(n-1)$ in the above result.

4. $(n+1) P_n(x) = P_{n+1}'(x) - x P_n'(x)$

proof:- From recurrence relations II & III, we have

$$n P_n(x) = x P_n'(x) - P_{n-1}'(x) \quad \text{—} (1)$$

and $(2n+1) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x) \quad \text{—} (2)$

subtracting (1) from (2), we get

$$(2n+1) P_n(x) - n P_n(x) = P_{n+1}'(x) - \cancel{P_{n-1}'(x)} - x P_n'(x) + \cancel{P_{n-1}'(x)}$$

$$\boxed{(n+1) P_n(x) = P_{n+1}'(x) - x P_n'(x)}$$

5. $(1-x^2) P_n'(x) = n\left[P_{n-1}(x) - x P_n(x)\right]$

Proof:- From recurrence relation II & IV, we have

$n P_n(x) = x P_n'(x) - P_{n-1}'(x)$ — ①

and $(n+1) P_n(x) = P_{n+1}'(x) - x P_n'(x)$ — ②

Replacing n by $(n-1)$ in ②, we get

$$(n-1+1) P_{n-1}(x) = P_{n-1+1}'(x) - x P_{n-1}'(x)$$

$$n P_{n-1}(x) = P_n'(x) - x P_{n-1}'(x) \quad — ③$$

multiplying ① by x, we get

$$n x P_n(x) = x^2 P_n'(x) - x P_{n-1}'(x) \quad — ④$$

subtract ④ from ③, we get

$$n P_{n-1}(x) - n x P_n(x) = P_n'(x) - \cancel{x P_{n-1}'(x)} - x^2 P_n'(x) + \cancel{x P_{n-1}'(x)}$$

$$= (1-x^2) P_n'(x)$$

$$\Rightarrow \boxed{n\left[P_{n-1}(x) - x P_n(x)\right] = (1-x^2) P_n'(x)}$$

6. $(1-x^2) P_n'(x) = (n+1)\left[x P_n(x) - P_{n+1}(x)\right]$

proof:- From recurrence relation IV, we have

$$(n+1) P_n(x) = P_{n+1}'(x) - x P_n'(x) \quad — ①$$

Replacing n by $(n-1)$ in eqn ①, we have

$$(n-1+1) P_{n-1}(x) = P_{n-1+1}'(x) - x P_{n-1}'(x)$$

$$n P_{n-1}(x) = P_n'(x) - x P_{n-1}'(x) \quad — ②$$

multiplying ② by x, we get

$$n x P_{n-1}(x) = x P_n'(x) - x^2 P_{n-1}'(x) \quad — ③$$

From recurrence relation II, we have

$$n P_n(x) = x P_n'(x) - P_{n-1}'(x) \quad — ④$$

Subsbracting (4) from (3), we get

$$nx P_{n-1}(x) - nP_n(x) = x\cancel{P_n'(x)} - x^2 P_{n-1}'(x) - x\cancel{P_n'(x)} + P_{n-1}'(x)$$

$$= -x^2 P_{n-1}'(x) + P_{n-1}'(x)$$

$$n[x P_{n-1}(x) - P_n(x)] = (1-x^2) P_{n-1}'(x) \quad \text{--- (5)}$$

Replacing n by $(n+1)$ in eqn (5), we have

$$(1-x^2) P_{n+1-1}'(x) = (n+1)[x P_{n+1-1}(x) - P_{n+1}(x)]$$

$$\boxed{(1-x^2) P_n'(x) = (n+1)[x P_n(x) - P_{n+1}(x)]}$$

<u>Beltrami's Result:</u>

To prove that

$$(2n+1)(x^2-1)P_n'(x) = n(n+1)[P_{n+1}(x) - P_{n-1}(x)]$$

(or) $$(2n+1)(1-x^2) P_n'(x) = n(n+1)[P_{n-1}(x) - P_{n+1}(x)]$$

<u>proof:</u>- we know that,

The recurrence relation V and VI are

$$(1-x^2) P_n'(x) = n[P_{n-1}(x) - x P_n(x)] \quad \text{--- (1)}$$

$$(1-x^2) P_n'(x) = (n+1)[x P_n(x) - P_{n+1}(x)] \quad \text{--- (2)}$$

From (1), we have

$$P_{n-1}(x) - x P_n(x) = \frac{1}{n}[(1-x^2) P_n'(x)]$$

$$\Rightarrow x P_n(x) = P_{n-1}(x) - \frac{1}{n}(1-x^2) P_n'(x)$$

substitute this value in eqn (2), we get

$$(1-x^2) P_n'(x) = (n+1)\left[P_{n-1}(x) - \frac{1}{n}(1-x^2) P_n'(x) - P_{n+1}(x)\right]$$

$$= (n+1) P_{n-1}(x) - \frac{n+1}{n}(1-x^2) P_n'(x) - (n+1) P_{n+1}(x)$$

$$(1-x^2)P_n'(x) + \frac{n+1}{n}(1-x^2)P_n'(x) = (n+1)P_{n-1}(x) - (n+1)P_{n+1}(x)$$

$$(1-x^2)\left[1+\frac{n+1}{n}\right]P_n'(x) = (n+1)\left[P_{n-1}(x) - P_{n+1}(x)\right]$$

$$(1-x^2)\left(\frac{n+n+1}{n}\right)P_n'(x) = -(n+1)\left[P_{n+1}(x) - P_{n-1}(x)\right]$$

$$(1-x^2)\left(\frac{2n+1}{n}\right)P_n'(x) = -(n+1)\left[P_{n+1}(x) - P_{n-1}(x)\right]$$

$$(2n+1)(1-x^2)P_n'(x) = -n(n+1)\left[P_{n+1}(x) - P_{n-1}(x)\right]$$

$$-(2n+1)(1-x^2)P_n'(x) = n(n+1)\left[P_{n+1}(x) - P_{n-1}(x)\right]$$

$$(2n+1)(x^2-1)P_n'(x) = n(n+1)\left[P_{n+1}(x) - P_{n-1}(x)\right]$$

which is known as Beltrami's result.

Christoffel's expansion:

$$P_n' = (2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n-5} + \cdots$$

the last term of the series being $3P_1$ or P_0 according as n is even or odd.

Expansion of an arbitrary function in a series of Legendre polynomials

Let $f(x)$ be a function defined in the interval $(-1,1)$. Assume that it is possible to expand $f(x)$ in a series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

$$= c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + \cdots \rightarrow (1)$$

To obtain the general coefficient, multiply on both sides of (1) by $P_n(x)$.

$$f(x)P_n(x) = c_0P_0(x)P_n(x) + c_1P_1(x)P_n(x) + \cdots + c_nP_n(x)P_n(x) + \cdots$$

$x = -1$ to 1, we get

$$\int_{-1}^{1} f(x) P_n(x)\,dx = \int_{-1}^{1} c_0 P_0(x) P_n(x)\,dx + \int_{-1}^{1} c_1 P_1(x) P_n(x)\,dx$$

$$+ \cdots + \int_{-1}^{1} c_n P_n^2(x)\,dx + \cdots$$

since $\int_{-1}^{1} P_m(x) P_n(x)\,dx = 0$, when $m \neq n$

$$= \frac{2}{2n+1}, \text{ when } m = n$$

$$\therefore \int_{-1}^{1} f(x) P_n(x)\,dx = 0 + 0 + \cdots + c_n \frac{2}{2n+1} + 0 + \cdots$$

$$= c_n \frac{2}{2n+1}$$

Hence $c_n = \dfrac{2n+1}{2n} \displaystyle\int_{-1}^{1} f(x) P_n(x)\,dx$.

which is a general coefficient in the expansion.

Bessel functions

Legendre polynomials :-

we know the Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \longrightarrow (1)$$

putting $n=0$ in (1), we get $P_0(x)=1$

putting $n=1$ in (1), we get $P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x$

putting $n=2$ in (1), we get

$$P_2(x) = \frac{1}{8}\frac{d^2}{dx^2}(x^2-1)^2$$

$$= \frac{1}{8}\frac{d}{dx}[2(x^2-1)2x]$$

$$= \frac{1}{2}\frac{d}{dx}(x^3-x)$$

$$= \frac{1}{2}(3x^2-1)$$

putting $n=3$ in (1), we get

$$P_3(x) = \frac{1}{48}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{48}\frac{d^2}{dx^2}[3(x^2-1)^2 \cdot 2x]$$

$$= \frac{1}{8}\frac{d}{dx}[(x^2-1)^2 + x\cdot 2(x^2-1)\cdot 2x]$$

$$= \frac{1}{8}\frac{d}{dx}(5x^4-6x^2+1)$$

$$= \frac{1}{8}(20x^3-12x)$$

$$= \frac{1}{2}(5x^3-3x)$$

Similarly, $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

and $P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$

Note:- These formulae can also be derived directly from the series expansion for $P_n(x)$.

① Show that $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{3x^2-1}{2}$ and hence express $2x^2-4x+2$ as a Legendre polynomial

Sol:- we know that Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \rightarrow ①$$

putting $n=0$ in (1), we get $P_0(x)=1$

putting $n=1$ in ①, we get $P_1(x)=\frac{1}{2}\frac{d}{dx}(x^2-1)=x$

putting $n=2$ in ①, we get

$$P_2(x) = \frac{1}{8}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}\left[2(x^2-1)\cdot 2x\right]$$

$$= \frac{1}{2}\frac{d}{dx}(x^3-x)$$

$$= \frac{1}{2}(3x^2-1) \rightarrow ②$$

From ②, we have

$$x^2 = \frac{1}{3}\left[2P_2(x)+1\right]$$

$$= \frac{1}{3}\left[2P_2(x)+P_0(x)\right] \quad [\because P_0(x)=1]$$

$$\therefore 2x^2-4x+2 = \frac{2}{3}\left[2P_2(x)+P_0(x)\right]-4P_1(x)+2P_0(x)$$

$$= \frac{4}{3}\left[P_2(x)-3P_1(x)+2P_0(x)\right]$$

② Show that $x^3 = \frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)$

Sol:- Since $P_3(x) = \frac{1}{2}(5x^3-3x)$

$$\therefore 2P_3(x) = 5x^3-3x$$

(or) $5x^3 = 2P_3(x)+3x$

(or) $x^3 = \frac{2}{5}P_3(x)+\frac{3}{5}x$

$$= \frac{2}{5}P_3(x)+\frac{3}{5}P_1(x) \quad [\because P_1(x)=x$$

$\Rightarrow$ prove that $x^4 = \frac{8}{35} P_4(x) + \frac{4}{7} P_2(x) + \frac{1}{5} P_0(x)$

Sol:- We know that

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

i.e $8P_4(x) = 35x^4 - 30x^2 + 3$

i.e $35x^4 = 8P_4(x) + 30x^2 - 3$

(or) $x^4 = \frac{8}{35} P_4(x) + \frac{30}{35} x^2 - \frac{3}{35}$

Since $P_2(x) = \frac{1}{2}(3x^2 - 1)$, we have

$2P_2(x) = 3x^2 - 1$

i.e $3x^2 = 2P_2(x) + 1$

(or) $x^2 = \frac{2}{3} P_2(x) + \frac{1}{3}$

$\Rightarrow$ Express $x^3 + 2x^2 - x - 3$ in terms of Legendre polynomials

Sol:- Let $f(x) = x^3 + 2x^2 - x - 3$

Since $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$\therefore x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} x$

Hence $f(x) = \left[\frac{2}{5} P_3(x) + \frac{3}{5} x\right] + 2x^2 - x - 3$

$= \frac{2}{5} P_3(x) + 2x^2 - \frac{2}{5} x - 3$

Since $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\therefore x^2 = \frac{2}{3} P_2(x) + \frac{1}{3}$

$\therefore f(x) = \frac{2}{5} P_3(x) + 2\left[\frac{2}{3} P_2(x) + \frac{1}{3}\right] - \frac{2}{5} x - 3$

$= \frac{2}{5} P_3(x) + \frac{4}{3} P_2(x) - \frac{2}{5} x - \frac{7}{3}$

$= \frac{2}{5} P_3(x) + \frac{4}{3} P_2(x) - \frac{2}{5} P_1(x) - \frac{7}{3} P_0(x)$

$[\because x = P_1(x), 1 = P_0(x)]$

⇒ using Rodrigue's formula, prove that $\int_{-1}^{1} x^m P_n(x)\,dx = 0$ if $m<n$

Sol:- we have Rodrigue's formula that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\therefore \int_{-1}^{1} x^m P_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1} x^m \frac{d^n}{dx^n}(x^2-1)^n$$

$$= \frac{1}{2^n n!}\int_{-1}^{1} x^m\, d\left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]$$

$$= \frac{1}{2^n n!}\left\{\left[x^m \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{1} - \int_{-1}^{1} m x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx\right\}$$

(using Integration by parts)

The first term in the R.H.S becomes zero at both limits because $\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$ will contain x^2-1 as a factor.

Hence $\int_{-1}^{1} x^m P_n(x)\,dx = \frac{(-1)^n m}{2^n n!}\int_{-1}^{1} x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$

Integrating by parts $(n-1)$ times, we get

$$\int_{-1}^{1} x^m P_n(x)\,dx = \frac{(-1)^n m}{2^n n!}\int_{-1}^{1}(x^2-1)^n \frac{d^{n-1}}{dx^{n-1}}(x^{m-1})\,dx$$

$$= 0, \left[\text{since } \frac{d^{n-1}}{dx^{n-1}}(x^{m-1}) = 0 \text{ if } m<n\right]$$

$$\therefore \int_{-1}^{1} x^m P_n(x)\,dx = 0 \text{ if } m<n$$

Generating function for $P_n(x)$:—

Th:- To show that $P_n(x)$ is the coefficient of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$.

(or) To show that $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$

Bessel functions:-

Bessel's Expansion Definition:-

The differential Equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$ where 'n' +ve integer

is known as Bessel's Equation.

It's particular solutions are called Bessel functions of order 'n'

General Solution of Bessel's differential Equation:-

The general solution of Bessel's Equation is

$$y = c_1 J_n(x) + c_2 J_{-n}(x),$$ where c_1 & c_2 are arbitrary constants

and $$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r-n}}{r!\,\Gamma(-n+r+1)}$$

Recurrence Formulae for $J_n(x)$:-

1. $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

(or)

$$\frac{n}{x} J_n(x) - J_n'(x) = J_{n+1}(x)$$

Proof:- we know that $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)} \rightarrow (1)$

diff both sides w.r.to 'x', we have

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r-1}\left(\frac{1}{2}\right)}{r!\,\Gamma(n+r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r} \left(\frac{x}{2}\right)^{-1} \frac{1}{2}}{r!\,\Gamma(n+r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\cancel{2}}}{\frac{\cancel{2}}{x}\, r!\,\Gamma(n+r+1)}$$

$$x J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{n(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} + \sum_{n=0}^{\infty} \frac{(-1)^r 2r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} + \sum_{r=0}^{\infty} (-1)^r \frac{2r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= n J_n(x) + 0 + \sum_{r=1}^{\infty} \frac{(-1)^r 2r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= n J_n(x) + \sum_{r=1}^{\infty} \frac{(-1)^r \cancel{2r}}{\cancel{r}(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \left(\frac{x}{\cancel{2}}\right)$$ 15

$$= n J_n(x) + \sum_{r=1}^{\infty} \frac{x\,(-1)^r}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

put $r-1 = s \Rightarrow r = s+1$

Then
$$x J_n'(x) = n J_n(x) + \sum_{s=1}^{\infty} \frac{x\,(-1)^{s+1}}{s!\,\Gamma(n+s+1+1)} \left(\frac{x}{2}\right)^{n+2(s+1)-1}$$

$$= n J_n(x) + x \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{s!\,\Gamma(n+s+2)} \left(\frac{x}{2}\right)^{n+2s+1}$$

$$= n J_n(x) + x \sum_{s=1}^{\infty} \frac{(-1)^s (-1)}{s!\,\Gamma(n+s+2)} \left(\frac{x}{2}\right)^{n+2s+1}$$

$$= n J_n(x) - x \sum_{s=1}^{\infty} \frac{(-1)^s}{s!\,\Gamma\{(n+1)+s+1\}} \left(\frac{x}{2}\right)^{n+1+2s}$$

$$\boxed{x J_n'(x) = n J_n(x) - x J_{n+1}(x)}$$ [$\because$ by the defn of $J_n(x)$]

2. $x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$

(or) $\frac{n}{x} J_n(x) + J_n'(x) = J_{n-1}(x)$.

<u>proof</u>:- we know that $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)}$

diff both sides w.r.to 'x', we get

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{2}}{r!\,\Gamma(n+r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r} \left(\frac{x}{2}\right)^{-1} \frac{1}{2}}{r!\,\Gamma(n+r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)\left(\frac{x}{2}\right)^{n+2r} \frac{\cancel{x}}{x} \frac{1}{\cancel{x}}}{r!\, \Gamma(n+r+1)}$$

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)\left(\frac{x}{2}\right)^{n+2r}}{x\, r!\, \Gamma(n+r+1)}$$

$$x J_n'(x) = \sum_{r=0}^{\infty} \frac{(n+2r)(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\, \Gamma(n+r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r-n)}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r)}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} - \sum_{r=0}^{\infty} \frac{(-1)^r n}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r 2(n+r)}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} - n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cancel{2}(n+r)}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \left(\frac{x}{\cancel{2}}\right) - n J_n(x)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)\, x}{r!\, (n+r+1-1)\, \Gamma(n+r+1-1)} \left(\frac{x}{2}\right)^{n+2r-1} - n J_n(x)$$

$$\left[\because \Gamma(n) = (n-1)\, \Gamma(n-1)\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cancel{(n+r)}\, x}{r!\, \cancel{(n+r)}\, \Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1} - n J_n(x)$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \Gamma(n+r)} \left(\frac{x}{2}\right)^{(n-1)+2r} - n J_n(x)$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n-1+2r}}{r!\, \Gamma(n+r+1-1)} - n J_n(x).$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n-1+2r}}{r!\, \Gamma((n-1)+r+1)} - n J_n(x)$$

$$x J_n'(x) = x J_{n-1}(x) - n J_n(x)$$

$$\Rightarrow x J_n'(x) + n J_n(x) = x J_{n-1}(x)$$

(or) $J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x)$.

3. $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

proof:- we know that $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\, \Gamma(n+r+1)}$

diff on both sides w.r.to 'x', we get

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r-1} \left(\frac{1}{2}\right)}{r!\, \Gamma(n+r+1)}$$

$$2 J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+r+r)}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} + \sum_{r=0}^{\infty} \frac{(-1)^r r}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)}{r!\, (n+r+1-1)\Gamma(n+r+1-1)} \left(\frac{x}{2}\right)^{n+2r-1} + 0.$$

$$+ \sum_{r=1}^{\infty} \frac{(-1)^r r}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cancel{(n+r)}}{r!\,\cancel{(n+r)}\, \Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1} + \sum_{r=1}^{\infty} \frac{(-1)^{r-1}(-1)\, \cancel{r}}{\cancel{r}(r-1)!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0} \frac{(-1)}{r!\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1} + \sum_{r=1}^{\infty} \frac{(-1)^{r-1}(-1)}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\{(n-1)+r+1\}} \left(\frac{x}{2}\right)^{(n-1)+2r} + \sum_{s=0}^{\infty} \frac{(-1)^s(-1)}{s!\,\Gamma(n+s+1+1)} \left(\frac{x}{2}\right)^{n+2(s+1)-1}$$

$$\left[\because r-1 = s \Rightarrow r = s+1\right]$$

$$= \sum_{r=0}^{\infty} J_{n-1}(x) - \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma\{(n+1)+s+1\}} \left(\frac{x}{2}\right)^{n+1+2s}$$

$$= J_{n-1}(x) - J_{n+1}(x)$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$\boxed{J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]}$$

Aliter: we know that recurrence relations (1) & (2) as

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \quad \text{--- (1)}$$

$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x) \quad \text{--- (2)}$$

Adding (1) & (2), we get

$$2x J_n'(x) = \cancel{n J_n(x)} - x J_{n+1}(x) - \cancel{n J_n(x)} + x J_{n-1}(x)$$

$$= x J_{n-1}(x) - x J_{n+1}(x)$$

$$= x\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$\cancel{x} J_n'(x) = \frac{1}{2}\cancel{x}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$\boxed{J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]}$$

4. $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

proof: we know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

multiplying both sides by $2n$, we get

$$2n J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\, 2n}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r-2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} - \sum_{r=0}^{\infty} \frac{(-1)^r\, 2r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2(n+r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} - \sum_{r=1}^{\infty} \frac{(-1)^r\, 2r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2(n+r)}{r!\,(n+r+1-1)\,\Gamma(n+r+1-1)} \left(\frac{x}{2}\right)^{n+2r-1} \left(\frac{x}{2}\right) - \sum_{r=1}^{\infty} \frac{(-1)^r\, 2\cancel{r}}{\cancel{r}(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \left(\frac{x}{2}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, \cancel{2}\cancel{(n+r)}}{r!\,\cancel{(n+r)}\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1} \left(\frac{x}{\cancel{2}}\right) - \sum_{r=1}^{\infty} \frac{(-1)^r\, \cancel{2}}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{\cancel{2}}\right) \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, x}{r!\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1} - \sum_{r=1}^{\infty} \frac{(-1)^r\, x}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, x}{r!\,\Gamma(n+r+1-1)} \left(\frac{x}{2}\right)^{n+2r-1} - x \sum_{r=1}^{\infty} \frac{(-1)^{r-1}(-1)}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\{(n-1)+r+1\}} \left(\frac{x}{2}\right)^{n+2r-1} + x \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= x\, J_{n-1}(x) + x \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-2+1}$$

$$= x J_{n-1}(x) + x \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(n+(s+1)+1)} \left(\frac{x}{2}\right)^{n+2(r-1)+1} \quad \left[\because r-1=s \atop r=s+1\right]$$

$$= x J_{n-1}(x) + x \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma((n+1)+s+1)} \left(\frac{x}{2}\right)^{n+2s+1}$$

$$= x J_{n-1}(x) + x \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma\{(n+1)+s+1\}} \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$= x J_{n-1}(x) + x J_{n+1}(x)$$

$$\therefore 2n J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

$$\boxed{J_n(x) = \frac{x}{2n}\left[J_{n+1}(x) + J_{n-1}(x)\right]}$$

Aliter: we know recurrence relations in ① & ② as

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \quad \text{— ①}$$

$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x) \rightarrow ②.$$

subtracting, we get

$$\cancel{x J_n'(x)} - \cancel{x J_n'(x)} = n J_n(x) - x J_{n+1}(x) + n J_n(x) - x J_{n-1}(x)$$

$$0 = 2n J_n(x) - x\left[J_{n+1}(x) + J_{n-1}(x)\right]$$

$$2n J_n(x) = x\left[J_{n+1}(x) + J_{n-1}(x)\right]$$

$$\boxed{J_n(x) = \frac{x}{2n}\left[J_{n+1}(x) + J_{n-1}(x)\right]}$$

5. $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

proof:- LHS $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_n'(x) + J_n(x)\, n x^{n-1}$

$$= x^{n-1} x J_n'(x) + n x^{n-1} J_n(x)$$

$$= x^{n-1}\left[x J_n'(x) + n J_n(x)\right]$$

$$\left[\{-n\,\cancel{J_n(x)} + x J_{n-1}(x)\} + n\,\cancel{J_n(x)}\right]$$

$$\left[\because \text{Recurrence relation } ② \quad x J_n'(x) = -n J_n(x) + x J_{n-1}(x)\right]$$

$$= x^{n-1}\left[x J_{n-1}(x)\right]$$

$$= x^n J_{n-1}(x) \text{ RHS.}$$

Hence $\boxed{\dfrac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)}$

Aliter: we know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

multiplying both sides by $\left(\frac{x}{2}\right)^{n}$, we get

$$\left(\frac{x}{2}\right)^{n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n}$$

$$\frac{x^n}{2^n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n}$$

$$x^n J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2n+2r}}{r!\,\Gamma(n+r+1)}\, 2^n$$

diff both sides w.r.to x, we get

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \frac{d}{dx} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2n+2r}}{r!\,\Gamma(n+r+1)}\, 2^n$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r) \left(\frac{x}{2}\right)^{2n+2r-1} \left(\frac{1}{2}\right)}{r!\,\Gamma(n+r+1)}\, 2^n$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, \cancel{2}(n+r)\, 2^n \left(\frac{x}{2}\right)^{n} \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\cancel{2}}}{r!\,\Gamma(n+r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+r) \cancel{2^n} \frac{x^n}{\cancel{2^n}} \left(\frac{x}{2}\right)^{n+2r-1}}{r! \, (n+r+\cancel{1}-\cancel{1}) \, \Gamma(n+r+\cancel{1}-\cancel{1})} \quad [\because \Gamma(n) = (n-1)\Gamma(n-1)]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cancel{(n+r)} \left(\frac{x}{2}\right)^{n+2r-1} x^n}{r! \, \cancel{(n+r)} \, \Gamma(n+r)}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{(n-1)+2r}}{r! \, \Gamma(n+r+1-1)}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{(n-1)+2r}}{r! \, \Gamma\{(n-1)+r+1\}}$$

$$= x^n J_{n-1}(x) \quad [\because \text{by the defn of } J_n(x)]$$

$$\therefore \boxed{\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)}$$

*** 6. $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

Proof :- LHS $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = x^{-n} J_n'(x) + J_n(x)(-n)x^{-n-1}$

$$= x^{-n-1} x^{+1} J_n'(x) + J_n(x)(-n)x^{-n-1}$$

$$= x^{-n-1}\left[x J_n'(x) - n J_n(x)\right]$$

$$= x^{-n-1}\left[\{\cancel{n J_n(x)} - x J_{n+1}(x)\} - \cancel{n J_n(x)}\right]$$

[$\because$ By recurrence relation ①
$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$]

$$= x^{-n-1}\left[(-x) J_{n+1}(x)\right]$$

$$= -x^{-n} J_{n+1}(x) \text{ RHS}$$

$$\therefore \boxed{\frac{d}{dx} x^{-n} J_n(x) = -x^{-n} J_{n+1}(x)}$$

Alitel: we know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \frac{x^{n+2r}}{2^{n+2r}}}{r!\,\Gamma(n+r+1)}$$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{n+2r} r!\,\Gamma(n+r+1)}$$

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{n} x^{2r}}{2^{n+2r} r!\,\Gamma(n+r+1)}$$

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{n+2r} r!\,\Gamma(n+r+1)}$$

diff both sides w.r.to 'x', we get

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \frac{d}{dx} \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{n+2r} r!\,\Gamma(n+r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2r\, x^{2r-1}}{r!\, 2^{n+2r}\,\Gamma(n+r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^{r-1}(-1)\, 2\cancel{r}\, x^{-n}\, x^{n+2r-1}}{\cancel{r}(r-1)!\;\Gamma(n+r+1)\, 2^{n+2r}}$$

$$= -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^{r-1} x^{n+1+2(r-1)}}{(r-1)!\;\Gamma(n+r+1)\, \frac{2^{n+2r}}{2}}$$

$$= -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^{r-1} x^{n+1+2(r-1)}}{(r-1)!\;\Gamma(n+r+1)\, 2^{n+2r-1}}$$

$$= -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^{r-1} x^{n+1+2(r-1)}}{(r-1)!\;\Gamma(n+r+1)\, 2^{n+1+2(r-1)}}$$

$$= -x^{-n} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{n+1+2s}}{s!\, \Gamma(n+s+1+1)}, \quad \text{where } r-1=s \Rightarrow r=s+1$$

$$= -x^{-n} J_{n+1}(x) \quad [\because \text{ by the defn of } J_n(x)]$$

$$\boxed{\therefore \frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)}$$

Eg:- **** Prove that $J_n(x) = (-1)^n J_n(x)$, where n is a +ve integer

Sol: we know that

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r!\, \Gamma(-n+r+1)}$$

But $\Gamma(n)$ is not defined, if n is a negative integer.

$\therefore$ The terms in $J_{-n}(x) = 0$ till $-n+r+1 \geq 1$

$$\Rightarrow -n+r \geq 0$$

$$\Rightarrow -n \geq r$$

$$\Rightarrow n \leq r$$

$$\Rightarrow r \geq n$$

So, we can write $J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r!\, \Gamma(-n+r+1)}$

put $r = n+s$, we get

~~$J_{-n}(x) = \Sigma$~~ Limits Lower limit

when $r = n$; ~~then~~ $r = n+s$

$$\Rightarrow n = n+s$$

$$\Rightarrow \boxed{s=0}$$

upper limit ; when $r = \infty \Rightarrow \boxed{s=\infty}$.

$$\therefore J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} \left(\frac{x}{2}\right)^{-(n)+2(n+s)}}{(n+s)!\, \Gamma(-\cancel{n}+\cancel{n}+s+1)}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^n (-1)^s \left(\frac{x}{2}\right)^{n+2s}}{(n+s)!\ \Gamma(s+1)}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{n+2s}}{(n+s)!\ \Gamma(s+1)}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{n+2s}}{\Gamma(n+s+1)\ s!} \quad \left[\because \begin{array}{l} \Gamma(n+1) = n! \\ \Gamma(n+r+1) = (n+r)! \end{array}\right]$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{n+2s}}{s!\ \Gamma(n+s+1)}$$

$= (-1)^n J_n(x)$ [$\because$ by the defn of $J_n(x)$]

$\therefore \boxed{J_{-n}(x) = (-1)^n J_n(x)}$

Eg:-2) Prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$.

(or) Determine the value of $J_{1/2}(x)$.

Sol:- we know that $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\ \Gamma(n+r+1)}$

put $n = \frac{1}{2}$, we get

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{\frac{1}{2}+2r}}{r!\ \Gamma(\frac{1}{2}+r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{1/2} \left(\frac{x}{2}\right)^{2r}}{r!\ \Gamma(r+3/2)}$$

$$= \left(\frac{x}{2}\right)^{1/2} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r}}{r!\ \Gamma(r+\frac{3}{2})}$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}} \left[\frac{1}{\Gamma(\frac{3}{2})} - \frac{1}{1!\,\Gamma(\frac{5}{2})}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(\frac{7}{2})}\left(\frac{x}{2}\right)^4 - \cdots \right]$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}} \left[\frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})} - \frac{1}{\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}\left(\frac{x}{2}\right)^2 + \frac{1}{\cancel{2}\,\frac{5}{\cancel{2}}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}\left(\frac{x}{2}\right)^4 - \cdots \right]$$

$$\left[\because \Gamma(n) = (n-1)\Gamma(n-1)\right]$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}} \frac{1}{\Gamma(\frac{1}{2})} \left[2 - \frac{\cancel{4}}{3}\,\frac{x^2}{\cancel{4}} + \frac{\cancel{4}}{15}\,\frac{x^4}{\cancel{16}_4} - \cdots \right]$$

$$= \sqrt{\frac{x}{2}}\; \frac{1}{\Gamma(\frac{1}{2})} \left[2 - \frac{2x^2}{2\cdot 3} + \frac{2x^4}{2\times 15\times 4} - \cdots \right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}}\; \frac{1}{\sqrt{\pi}} \left[2 - \frac{2x^2}{3!} + \frac{2x^4}{2\times 3\times 4\times 5} - \cdots \right]$$

$$\because \left[\because \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}}\; \frac{1}{\sqrt{\pi}}\; \frac{x}{2}\; \frac{2}{x} \left[2 - \frac{2x^2}{3!} + \frac{2x^4}{5!} - \cdots \right]$$

$$= \frac{\cancel{\sqrt{x}}}{\sqrt{2}}\; \frac{1}{\sqrt{\pi}}\; \frac{x}{\cancel{2}}\; \frac{2}{(\sqrt{x})^{\cancel{2}}}\,\cancel{2} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots \right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi x}} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi x}} \sin x \quad \left[\because \text{sin series } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

Hence $J_{1/2}(x) = \dfrac{\sqrt{2}}{\sqrt{\pi x}} \sin x$

$$= \sqrt{\frac{2}{\pi x}} \sin x.$$

Generating Function for $J_n(x)$

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

Orthogonality of Bessel's Function:

Orthogonality of Bessel Function is

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ \frac{1}{2}[J_{n+1}(\alpha)]^2, & \text{if } \alpha = \beta \end{cases}$$

where α and β are the roots of $J_n(x) = 0$.

Proof:- WKT $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ are the solutions of Bessel equations

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2) u = 0 \longrightarrow (1)$$

and $$x^2 v'' + x v' + (\beta^2 x^2 - n^2) v = 0 \longrightarrow (2)$$

Multiplying eq (1) by $\frac{v}{x}$ and eq (2) $\frac{u}{x}$ and subtracting we get

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2) x uv = 0.$$

(or)

$$\frac{d}{dx}[x(u'v - v'u)] = (\beta^2 - \alpha^2) x uv$$

Now integrating both sides w.r.to 'x' from 0 to 1 we get

$$(\beta^2 - \alpha^2)\int_0^1 x uv\,dx = [x(u'v - uv')]_{x=0}^{1}$$

$$= [u'v - uv']_{x=1} \longrightarrow (3)$$

$\therefore$ $u = J_n(\alpha x)$ and $v = J_n(\beta x)$

Then $u' = J_n'(\alpha x)\cdot\alpha$ and $v' = J_n'(\beta x)\cdot\beta$

Substitute these values in eq (3)

we get

$$\int_0^1 x\cdot J_n(\alpha x)\, J_n(\beta x)\,dx = \frac{\alpha\cdot J_n'(\alpha)\, J_n(\beta) - \beta\cdot J_n(\alpha)\, J_n'(\beta)}{\beta^2-\alpha^2} \rightarrow (4)$$

Case (i): — let α and β are distinct roots of the eq $J_n(x)=0$ then $J_n(\alpha)=0$, $J_n(\beta)=0$

eq (4) becomes

$$\int_0^1 x\cdot J_n(\alpha x)J_n(\beta x)\,dx = 0, \quad \alpha\neq\beta$$

$\therefore$ The functions $J_n(\alpha x)$ and $J_n(\beta x)$ are orthogonal

Case (ii): — Let $\alpha=\beta$

Then the right side of eq (4) is of $\frac{0}{0}$ form

Considering 'α' as a root of $J_n(x)=0$ and β as a variable approaching 'α'

then eq (4) gives

$$\lim_{\beta\to\alpha}\int_0^1 x\cdot J_n(\alpha x)\cdot J_n(\beta x)\,dx = \lim_{\beta\to\alpha}\frac{\alpha\cdot J_n'(\alpha)\, J_n(\beta) - 0}{\beta^2-\alpha^2}$$

$$\text{i.e } \int_0^1 x\left[J_n(\alpha x)\right]^2 dx = \lim_{\beta\to\alpha}\frac{\alpha\cdot J_n'(\alpha)\, J_n'(\beta)}{2\cdot\beta} \quad \text{using L Hospital rul}$$

$$= \frac{1}{2}\left[J_n'(\alpha)\right]^2$$

$$\therefore \int_0^1 x\left[J_n(\alpha x)\right]^2 dx = \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2$$

From R.R. I

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

$$J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$$

$$J_n'(\alpha) = -J_{n+1}(\alpha)$$

prove that $e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

(or) show that the coefficient of t^n in the power series expansion of $e^{\frac{x}{2}(t-\frac{1}{t})}$ is $J_n(x)$.

(or) prove that when n is a positive integer $J_n(x)$ is the coefficient of z^n in the expansion of $e^{\frac{x}{2}(z-\frac{1}{z})}$ in ascending and descending powers of z.

(or) state and prove Generating function for $J_n(x)$.

sol:- we have $e^{\frac{x}{2}(t-\frac{1}{t})} = e^{xt/2} \times e^{-x/2t}$

$$= \left[1+\left(\frac{xt}{2}\right)+\frac{1}{2!}\left(\frac{xt}{2}\right)^2+\frac{1}{3!}\left(\frac{xt}{2}\right)^3+ \cdots\right]\left[1-\left(\frac{x}{2t}\right)+\frac{1}{2!}\left(\frac{x}{2t}\right)^2 \cdots\right]$$

The coefficient of t^n in this product = sum of the products of the coefficients of $t^n, t^{n+1}, t^{n+2}, \cdots$ in the 1st

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \cdots$$

$$= \frac{(-1)^0}{0!\,\Gamma(n+1)}\left(\frac{x}{2}\right)^n + \frac{(-1)^1}{1!\,\Gamma(n+2)}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{2!\,\Gamma(n+3)}\left(\frac{x}{2}\right)^{n+4} - \cdots$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} = J_n(x)$$

11^{ly} the coefficient of t^{-n} in this product

$$= \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^{n+2}}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} + \cdots$$

$$= (-1)^n\left[\frac{(-1)^0}{0!\,\Gamma(n+1)}\left(\frac{x}{2}\right)^n + \frac{(-1)^1}{1!\,\Gamma(n+2)}\left(\frac{x}{2}\right)^{n+2} + \cdots\right]$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} = (-1)^n J_n(x) = J_{-n}(x)$$

Finally the coefficient of t^0 in this product

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} - - - = J_0(x)$$

Hence

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = J_0(x) + t J_1(x) + t^2 J_2(x) + t^3 J_3(x)$$

$$+ - - + t^{-1} J_{-1}(x) + t^{-2} J_{-2}(x) + - -$$

$$= J_0(x) + \left(t - \frac{1}{t}\right) J_1(x) + \left(t^2 + \frac{1}{t^2}\right) J_2(x) + -$$

$$- - + \left(t^n + (-1)^n \frac{1}{t^n}\right) J_n(x) + - -$$

$$= \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

Bessel functions of various orders can be derived as coefficients of different powers of t in the expansion of $e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}$ is known as generating function of Bessel Equation.

$\Rightarrow$ Show that (a) $\cos x = J_0 - 2J_2 + 2J_4 - \cdots]$ $\quad t = e^{i\theta}$

(b) $\sin x = 2(J_1 - J_3 + J_5 - \cdots]$ $\quad$ and $\theta = \pi/2$

$\Rightarrow$ using Jacobi series, prove that $J_0^2 + 2[J_1^2 + J_2^2 + J_3^2 + \cdots] = 1$

Sol:- The Jacobi series are

$$\cos(x\sin\theta) = J_0 + 2(J_2\cos 2\theta + J_4\cos 4\theta + \cdots) \rightarrow (1)$$

and $\sin(x\sin\theta) = 2(J_1\sin\theta + J_3\sin 3\theta + \cdots) \rightarrow (2)$

Also we know that if m & n are integers

$$\int_0^\pi \sin m\theta \sin n\theta \, d\theta = \int_0^\pi \cos m\theta \cos n\theta \, d\theta = \begin{cases} \pi/2, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$$

Individually squaring (1) and integrating with respect to θ between the limits 0 to π, we get

$$\int_0^\pi \cos^2(x\sin\theta)\, d\theta = J_0^2\pi + 2J_2^2\pi + 2J_4^2\pi + \cdots \rightarrow (3)$$

Now squaring (2) and integrating w.r.to θ b/w the limits 0 to π, we get

$$\int_0^\pi \sin^2(x\sin\theta)\, d\theta = 2J_1^2\pi + 2J_3^2\pi + 2J_5^2\pi + \cdots \rightarrow (4)$$

(3) + (4) gives

$$\int_0^\pi [\cos^2(x\sin\theta) + \sin^2(x\sin\theta)]\, d\theta = \pi[J_0^2 + 2(J_1^2 + 2J_2^2 + J_3^2 + \cdots)]$$

(or) $\int_0^\pi d\theta = \pi[J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots)]$

(or) $(\theta)_0^\pi = \pi = \pi[J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots)]$

$$\therefore J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots) = 1$$

$\Rightarrow$ Show that

(i) $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta$, n being an integer.

(ii) $J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\, d\theta = \frac{1}{\pi}\int_0^\pi \cos(x\cos\theta)\, d\theta$

Sol:- (i) The Jacobi series are

$$\cos(x\sin\theta) = J_0 + 2(J_2\cos 2\theta + J_4\cos 4\theta + \cdots) \rightarrow (1)$$

$$\sin(x\sin\theta) = 2(J_1\sin\theta + J_3\sin 3\theta + \cdots) \rightarrow (2)$$

⇒ Show that

(a) $\cos(x\sin\theta) = J_0 + 2(J_2\cos 2\theta + J_4\cos 4\theta + \cdots)$

(b) $\sin(x\sin\theta) = 2(J_1\sin\theta + J_3\sin 3\theta + J_5\sin 5\theta + \cdots)$

Sol:- we know that the generating function of $J_n(x)$ is given by

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

$$= J_0 + J_1\left(t-\frac{1}{t}\right) + J_2\left(t^2-\frac{1}{t^2}\right) + J_3\left(t^3-\frac{1}{t^3}\right) + \cdots \rightarrow (1)$$

$[\because J_{-n}(x) = (-1)^n J_n(x)]$

Now put $t = \cos\theta + i\sin\theta$

So that $t^p = \cos p\theta + i\sin p\theta$

and $\frac{1}{t^p} = \cos p\theta - i\sin p\theta$

giving $t^p + \frac{1}{t^p} = 2\cos p\theta$

$t^p - \frac{1}{t^p} = 2i\sin p\theta$

Substituting these in (1), we get

$$e^{ix\sin\theta} = J_0 + 2[J_2\cos 2\theta + J_4\cos 4\theta + \cdots] + 2i[J_1\sin\theta + J_3\sin 3\theta + \cdots]$$

$$\cos(x\sin\theta) + i\sin(x\sin\theta) = J_0 + 2[J_2\cos 2\theta + J_4\cos 4\theta + \cdots] + 2i[J_1\sin\theta + J_3\sin 3\theta + \cdots] \rightarrow (2)$$

$[\because e^{i\theta} = \cos\theta + i\sin\theta$

Equating real and imaginary parts in (2), we get

$$\cos(x\sin\theta) = J_0 + 2[J_2\cos 2\theta + J_4\cos 4\theta + \cdots]$$

and $\sin(x\sin\theta) = 2[J_1\sin\theta + J_3\sin 3\theta + \cdots]$

These are known as Jacobi series.

H.W

⇒ Establish the Jacobi series:

(i) $\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \cdots]$

(ii) $\sin(x\cos\theta) = 2[J_1\cos\theta - J_3\cos 3\theta + J_5\cos 5\theta - \cdots]$ Replace θ by $\frac{\pi}{2}-\theta$

1. Prove that $J_{-n}(x) = (-1)^n J_n(x)$, where n is a positive integer.

Sol: we have $J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$

But $\Gamma(n)$ is not defined, if n is a negative integer.

$\therefore$ Terms in $J_n(x)$ are equal to zero till

$-n+r+1 \geq 1$ or $r \geq n$.

So, we can write $J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$

putting $r = n+s$, we get

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{n+s}}{(n+s)!\,\Gamma(-n+n+s+1)} \left(\frac{x}{2}\right)^{-n+2n+2s}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{(n+s)!\,\Gamma(s+1)} \left(\frac{x}{2}\right)^{n+2s}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(n+s+1)} \left(\frac{x}{2}\right)^{n+2s}$$

$\because \Gamma(s+1) = s!,\ (n+s)! = \Gamma(n+s+1)$

$$\therefore \boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

Prove that $J_n(-x) = (-1)^n J_n(x)$, where n is a positive negative integer

we have $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)} \longrightarrow (1)$

case(i): If n is a positive integer.

put $x = -x$ in ①, we have

$$J_n(-x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(-\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} (-1)^{n+2r} \left(\frac{x}{2}\right)^{n+2r}$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

$$J_n(-x) = (-1)^n J_n(x)$$

case(ii): If n is a negative integer.

Let $n = -m$, where m is a positive integer.

Then, $J_n(x) = J_{-m}(x) = (-1)^m J_m(x)$

put $x = -x$ in the above relation, we get

$$J_n(-x) = J_{-m}(-x) = (-1)^m J_m(-x)$$

$$= (-1)^m \left[(-1)^m J_m(x)\right], \text{ from case(i)}$$

$$= (-1)^{2m} J_m(x)$$

$$= J_m(x)$$

$$= J_{-n}(x)$$

$$= (-1)^n J_n(x).$$

Hence $J_n(-x) = (-1)^n J_n(x)$, where n is a positive or negative integer.

(3) Show that $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{1}{x}\sin x - \cos x\right]$

Sol: we have $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$ — ①

put $n = \frac{1}{2}$, we have

$$J_{1/2}(x) = \frac{x}{2\frac{1}{2}}\left[J_{\frac{1}{2}-1}(x) + J_{\frac{1}{2}+1}(x)\right]$$

$$= x\left[J_{-1/2}(x) + J_{3/2}(x)\right]$$

(or) $x J_{\frac{3}{2}}(x) = J_{\frac{1}{2}}(x) - x J_{-1/2}(x)$

$$= \sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\, x\cos x$$

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{1}{x}\sin x - \cos x\right]$$

4) Show that $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)$

Sol:- we have $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

put $n = 1, 2, 3, 4$ successively, we obtain

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x) \quad — ①$$

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x) \quad — ②$$

$$J_4(x) = \frac{6}{x}J_3(x) - J_2(x) \rightarrow ③$$

$$J_5(x) = \frac{8}{x}J_4(x) - J_3(x) \quad — ④$$

substitute the value of $J_2(x)$ in eqn ②, we get

$$J_3(x) = \frac{4}{x}\left[\frac{2}{x}J_1(x) - J_0(x)\right] - J_1(x)$$

$$= \frac{8}{x^2}J_1(x) - \frac{4}{x}J_0(x) - J_1(x)$$

$$= \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x) \quad \text{— ⑤}$$

put eqn ⑤, eqn ① in eqn ③, we get

$$J_4(x) = \frac{6}{x}\left\{\left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)\right] - \left[\frac{2}{x}J_1(x) - J_0(x)\right]$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x).$$

④ prove that $\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$

Sol: $\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = J_n(x) J_{n+1}(x) + x\left[J_n(x) J'_{n+1}(x) + J_n'(x) J_{n+1}(x)\right]$

$$= J_n(x) J_{n+1}(x) + \left[x J_n'(x)\right] J_{n+1}(x) + \left[x J'_{n+1}(x)\right] J_n(x) \quad \text{①}$$

we know that

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \quad \text{— ②}$$

and $x J_n'(x) = -n J_n(x) + x J_{n-1}(x) \rightarrow$ ③

Replacing n in ③ by n+1, we have

$$x J'_{n+1}(x) = -(n+1) J_{n+1}(x) + x J_n(x) \rightarrow \text{④}$$

Now putting the values of $x J_n'(x)$ and $x J'_{n+1}(x)$ from ③ and ④ in ①, we obtain

$$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = J_n(x) J_{n+1}(x) + \left[n J_n(x) - x J_{n+1}(x)\right] J_{n+1}(x)$$
$$+ \left[-(n+1) J_{n+1}(x) + x J_n(x)\right] J_n(x)$$

$$= J_n(x) J_{n+1}(x) + n J_n(x) J_{n+1}(x) - x J^2_{n+1}(x) - (n+1) J_{n+1}(x) J_n(x)$$
$$+ x J_n^2(x)$$

$$= (n+1) J_n(x) J_{n+1}(x) - x J^2_{n+1}(x) - (n+1) J_{n+1}(x) J_n(x) + x J_n^2(x)$$

$$= -x J^2_{n+1}(x) + x J_n^2(x)$$

$$= x\left[J_n^2(x) - J^2_{n+1}(x)\right].$$

(5) Prove that $J_0^2 + 2(J_1^2 + J_2^2 + --) = 1$

sol: we have $\frac{d}{dx}\left[J_n^2 + J^2_{n+1}\right] = \frac{2}{x}\left[n J_n^2 - (n+1) J^2_{n+1}\right]$ — (1)

put $n = 0, 1, 2, --$ successively in (1), we have

$$\frac{d}{dx}\left[J_0^2 + J_1^2\right] = 2\left[0 - \frac{1}{x} J_1^2\right]$$

$$\frac{d}{dx}\left[J_1^2 + J_2^2\right] = 2\left[\frac{1}{x} J_1^2 - \frac{2}{x} J_2^2\right]$$

$$\frac{d}{dx}\left[J_2^2 + J_3^2\right] = 2\left[\frac{2}{x} J_2^2 - \frac{3}{x} J_3^2\right]$$

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Adding these column wise and remembering that $J_n(x) \to 0$ as $n \to \infty$, we get

$$\frac{d}{dx}\left[J_0^2 + 2(J_1^2 + J_2^2 + --)\right] = 0$$

Integrating w.r.to x, we get

$$J_0^2 + 2(J_1^2 + J_2^2 + --)] = C$$; where c is a constant — (2)

Also we know that $J_0(0) = 1$ and $J_n(0) = 0$ for $n \geq 1$

$\therefore$ From (2), put $x = 0$, we have $1 + 2(0) = C$

$$\Rightarrow \boxed{C = 1}$$

$\therefore$ From (2), $J_0^2 + 2(J_1^2 + J_2^2 + \cdots) = 1$.

Unit III Complex Function

Functions of complex variable

Introduction:-

① complex Number (or) complex variable:-

A complex number z is an order pair (x,y) of real numbers and is written as $z=x+iy$ where $x,y\in R$ and $i=\sqrt{-1}$ and x is real part of z, y is imaginary part of z

If x,y are real variables then $z=x+iy$ is called complex variable.

② polar form of a complex Number:-

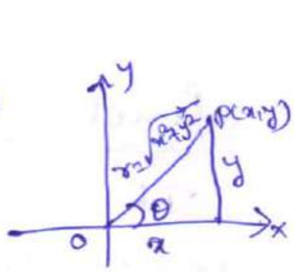
Let $z=x+iy$

where $x=r\cos\theta$, $y=r\sin\theta$

Then $z=r(\cos\theta+i\sin\theta)=re^{i\theta}$ is called polar form of a complex number

$r=\sqrt{x^2+y^2}$ is called Modulus of z and it is denoted by $|z|$

$\therefore |z|=\sqrt{x^2+y^2}$



$\theta=\tan^{-1}\left(\frac{y}{x}\right)$ is called amplitute (or) argument of z (or) $\arg z$

Complexplane

③ conjugate of a Complex Number:-

If $z=x+iy$ then $x-iy$ is called complex conjugate of z and it is denoted by $\bar{z}$

i.e If $z=x+iy$ then $\bar{z}=x-iy$

properties:- ① $z\bar{z}=|\bar{z}|^2=x^2+y^2$

② If $z=\bar{z}$ if z is real

③ $z+\bar{z}=2\operatorname{Re} z$

④ $z-\bar{z}=2\operatorname{Im} z$

⑤ $\overline{z_1\pm z_2}=\bar{z}_1\pm\bar{z}_2$

⑥ $\overline{z_1 z_2}=\bar{z}_1\cdot\bar{z}_2$

⑦ $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}$

Important operators:-

① Equality:- let $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$ then $z_1=z_2 \Rightarrow x_1=x_2$; $y_1=y_2$

② Addition & subtraction:- $z_1+z_2 = (x_1+x_2)+i(y_1+y_2)$

$z_1-z_2 = (x_1-x_2)+i(y_1-y_2)$

③ Multiplication:- $z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

④ Division:- $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

properties of Argument:-

① $\text{Arg } z_1 z_2 = \theta_1+\theta_2 = \arg z_1 + \arg z_2$

② $\text{Arg } \frac{z_1}{z_2} = \theta_1-\theta_2 = \arg z_1 - \arg z_2$

Properties of Modulus:- ① $|z_1+z_2| \leq |z_1|+|z_2|$

② $|z_1-z_2| \geq |z_1|-|z_2|$

③ $|z_1 z_2| = |z_1||z_2|$

④ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

function of complex variable (or) complex function:-

If $z=x+iy$ is a complex variable then function of a complex variable is denoted by w (or) $f(z)$ and is defined as

$$w = f(z) = u(x,y) + iv(x,y)$$

where u,v are function of real variables x,y and u,v are called real and imaginary parts of $f(z)=w$

The image of z under the function f is $w=f(z)$ this is also a complex number.

Ex:- If $w=f(z)=z^2+z$ find its real and imaginary parts also find $f(z)$ at $1+i$

Sol:- we know that $z=x+iy$

Now $w=f(z) = z^2+z$

$= (x+iy)^2+(x+iy)$

$= (x^2-y^2+x)+i(2xy+y) = u+iv$

where $u=x^2-y^2+x$

$v = 2xy+y$

Now $f(z)_{\text{at } 1+i} = f(1+i) = (1+i)^2+(1+i)$

$= 1+3i$

Roots of complex numbers:-

(i) $(\cos\theta+i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)$ is an n^{th} root of the expression $\cos\theta+i\sin\theta$ i.e a value of $(\cos\theta+i\sin\theta)^{1/n}$, for an integral value of k.

Limit:- 'w_0' is said to be limit of $f(z)$ at $z=z_0$ if for every $\epsilon>0$ there exists $\delta>0$ such that $|f(z)-w_0|<\epsilon$ whenever $0<|z-z_0|<\delta$.

and it is denoted by $\lim_{z\to z_0} f(z)=w_0$ (2)

Ex:- Find $\lim_{z\to 1+i} z^2-5z+10$

Sol:- $\lim_{z\to 1+i} z^2-5z+10 = \lim_{z\to 1+i} z^2 - \lim_{z\to 1+i} 5z + \lim_{z\to 1+i} 10$

$= (1+i)^2 - 5(1+i) + 10$

$= 5-3i$

Continuity:- A function $f(z)$ is said to be continuous at $z=z_0$ if $f(z_0)$ exists, $\lim_{z\to z_0} f(z)$ exists and $\lim_{z\to z_0} f(z) = f(z_0)$

. A function is said to be continuous in a Domain if it is continuous at every point of the domain.

Note:- If $f(z_0)$ does not exists or $\lim_{z\to z_0} f(z)$ does not exists (or) $\lim_{z\to z_0} f(z) \neq f(z_0)$ then the function $f(z)$ is not continuous at $z=z_0$ (or) $f(z)$ is discontinuous at $z=z_0$

Differentiability:-

Let $w=f(z)$ be a complex function Then the derivative of $f(z)$ at the point z_0 is denoted by $f'(z_0)$ and it is defined as

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

(OR) If $\boxed{z-z_0=\Delta z}$ then

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

Theorem:- If $f(z)$ is differentiable at a point z_0 then it is continuous at that point

Proof:- Given $f(z)$ is differentiable at a point z_0

i.e $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \rightarrow (1)$

$\therefore f(z_0)$ is well defined.

Now let $\lim_{z\to z_0} (f(z)-f(z_0)) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{(z-z_0)} \times (z-z_0)$

$= \left[\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}\right] \cdot \lim_{z\to z_0} (z-z_0)$

$= f'(z_0)\cdot 0 = 0 \Rightarrow \lim_{z\to z_0} f(z)-f(z_0) = 0$

$\therefore$ $f(z_0)$ is continuous at z_0 $\quad \lim_{z\to z_0} f(z) = f(z_0)$

① Show that the function $f(z)=z^n$, where n is +ve integer. is differentiable for all values of z.

Sol:- Given $f(z)=z^n$ where $n\in z^+$

By defintion of derivatives

WKT $f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$= \lim_{\Delta z\to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z}$

$= \lim_{\Delta z\to 0} \frac{\cancel{z^n} + n z^{n-1}\Delta z + nc_2 z^{n-2}(\Delta z)^2 + \cdots + (\Delta z)^n - \cancel{z^n}}{\Delta z}$

$= \lim_{\Delta z\to 0} \frac{\cancel{\Delta z}[n z^{n-1} + nc_2 z^{n-2}(\Delta z) + \cdots + (\Delta z)^{n-1}]}{\cancel{\Delta z}}$

$\therefore \boxed{f'(z) = n z^{n-1}}$

Hence $f'(z)$ exists for all values of z

② using the definition, derivative find the derivative of z^2 for all z.

③ Find the derivative of $w=f(z) = z^3-2z$ at the point

(i) $z=z_0$ (ii) $z=1$

④ Show that the function $f(z) = \frac{\bar{z}}{z}$ is not continuous at $z=0$

Sol:- Let $z=x+iy$

suppose $z\to 0$ along x-axis, then we have $y=0$

i.e $z=x$; $\bar{z}=x$

$\therefore \lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{x\to 0} \frac{x}{x} = 1$

Again suppose $z \to 0$ along y-axis then $x=0$

i.e $z = iy$; $\bar{z} = -iy$ (3)

$$\therefore \underset{z\to 0}{Lt} \frac{\bar{z}}{z} = \underset{iy\to 0}{Lt} \frac{-iy}{iy} = -1$$

$$\therefore \underset{z\to 0}{Lt} \frac{\bar{z}}{z} \text{ does not exists}$$

{ Left limit = Right limit = function at the point
(or)
function along x-axis = function along y-axis is called continuous }

(5) If $f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+x^2}, & z_0 \neq 0 \\ 0, & z_0 = 0 \end{cases}$ is not differentiable at origin.

Sol:- The given function

$$f(z) = \frac{x^3y(y-ix)}{x^6+x^2} \Rightarrow f(0) = 0$$

$$\text{But } f'(0) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z-0}$$

$$= \underset{z\to 0}{Lt} \frac{\left(\frac{x^3y(y-ix)}{x^6+x^2}\right) - 0}{(x+iy) - 0}$$

$$= \underset{z\to 0}{Lt} \frac{x^3y(y-ix)}{(x^6+x^2)(x+iy)}$$

suppose $z \to 0$ along the path $y \to x$
& $x \to 0$

$$f'(0) = \underset{\substack{y\to x \\ x\to 0}}{Lt} \frac{x^3y(y-ix)}{(x^6+x^2)(x+iy)}$$

$$= \underset{x\to 0}{Lt} \frac{x^4(x-ix)}{(x^6+x^2)(x+ix)} = \underset{x\to 0}{Lt} \frac{x^5(1-i)}{(x^6+x^2)x(1+i)} = \underset{x\to 0}{Lt} \frac{x^{\cancel{8}\,2}(1-i)}{x^3(x^4+1)(1+i)}$$

$$= \underset{x\to 0}{Lt} \frac{x^2(1-i)}{(x^4+1)(1+i)} = 0$$

Suppose $z \to 0$ along the path $y \to x^2$ & $x \to 0$

$$f'(0) = \lim_{\substack{y \to x^2 \\ x \to 0}} \frac{x^3 y (y - ix)}{(x^6+y^2)(x+iy)}$$

$$= \lim_{x \to 0} \frac{x^3 x^2 (x^2 - ix)}{(x^6+x^4)(x+ix^2)}$$

$$= \lim_{x \to 0} \frac{x^6 (x-i)}{x^5 (x^2+1)(1+ix)}$$

$$= 0.$$

Suppose $z \to 0$ along the path $y \to x^3$ & $x \to 0$

$$f'(0) = \lim_{\substack{y \to x^3 \\ x \to 0}} \frac{x^3 y (y - ix)}{(x^6+y^2)(x+iy)}$$

$$= \lim_{x \to 0} \frac{x^3 \cdot x^3 (x^3 - ix)}{(x^6+x^6)(x+ix^3)}$$

$$= \lim_{x \to 0} \frac{x^7 (x^2 - i)}{2x^7 (1+ix^2)}$$

$$= \frac{-i}{2}$$

Since we are getting different limits for different paths. Hence the given function is not differentiable at the origin.

(6) Prove that the function $f(z)$ defined by

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is continuous at the origin.

Sol:- Given $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$

then

$$\lim_{z \to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \lim_{y \to 0} \frac{-y^3(1-i)}{y^2} = 0$$

and

$$\lim_{z \to 0} f(z) = \lim_{\substack{y \to 0 \\ x \to 0}} f(z) = \lim_{\substack{y \to 0 \\ x \to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \lim_{x \to 0} \frac{x^3(1+i)}{x^2} = 0$$

Also

$f(0) = 0$ by the given data.

Thus, we get $\lim_{z \to 0} f(z) = f(0)$

Now take both x & y simultaneously along the path $y = mx$, then

$$\lim_{z \to 0} f(z) = \lim_{\substack{y \to mx \\ x \to 0}} \frac{x^3(1+i) - m^3 x^3 (1-i)}{x^2 + x^2 m^2} = \lim_{x \to 0} \frac{x^3((1+i) - m^3(1-i))}{x^2(1+m^2)}$$

$$= 0$$

$\therefore$ what ever the manner in which $z \to 0$, we have

$$\underset{z\to 0}{Lt} f(z) = 0 = f(0)$$

(7)

$\therefore$ The function is continuous at $z=0$

Note:- ① To solve the problems on continuity (or) Differentiability take the three paths given below

(i) $y \to mx$ and $x \to 0$
$y \to x$ and $x \to 0$
(ii) $y \to x^2$ and $x \to 0$
(iii) $y \to x^3$ and $x \to 0$

x, x^2, x^3 my

② Every differentiable functions is continuous and converse not true

Analytic Functions:-

A function $f(z)$ is said to be analytic at a point z_0, if f is differentiable not only at z_0 but at every point of some neighbourhood of z_0.

An analytic function is also known as "homomorphic" and "Regular" (or) "monogenic"

Entire functions:- If $f(z)$ is analytic at every point z on the complex plane then $f(z)$ is called Entire function (or) integral function

Ex:- polynomial functions are entire functions

Note:- If $f(z)$ is analytic at a point z_0

(i) $f'(z_0)$ exists and

(ii) $f'(z)$ exists at every point z in a nbhd of z_0

Circles:- $|z-a|=r$ represents a circle with center at the point 'a' and radious 'r'

Neighbourhood of a point:-

The Set of all points in the distance of a smallest +ve integer ϵ from the point at z_0 is called nbd

$|z-z_0|=\epsilon$

nbd

Deleted nbhd of a point:-

The Set $\{z / 0 < |z-z_0| < \delta\}$ is called the deleted nbhd of z_0

Singular point:- A point $z=z_0$ is said to be singular point of $f(z)$ if it is not analytic at z_0.

Note:- To find the singular points, equate the denominator to zero and solve for z.

Ex:- (1) For the function $f(z)=\frac{1}{z-a}$, $z=a$ is a Singular point

(2) $f(z)=\frac{z^2}{(z-3)(z-4)} \Rightarrow z=3, z-4$ are singular points

Isolated singular points:- Suppose $z=a$ is a Singular point of a function $f(z)$ and no other Singular point of $f(z)$ exists in a circle with Centre at 'a'. Then $z=a$ is said an "Isolated Singular point" of $f(z)$

Ex:- $f(z)=\frac{1}{z}$ is analytic at every point $z \neq 0$

$f'(z) = \frac{-1}{z^2}$, if $z \neq 0$

At $z=0$, $f'(z)$ doesnot exists $\therefore$ $z=0$ is an Isolated singular point of $f(z)$

Cauchy - Riemann equation in cartesian co-ordinates (C-R equations)

Statement:-

The necessary and sufficient condition for the derivative of the function $f(z) = w = u(x,y) + i v(x,y)$ to exist for all values of z in domain R are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x & y in R

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ i.e $u_x = v_y$; $u_y = -v_x$

The above relation are known as C-R equations

(or)

Derive the necessary & sufficient condition for $f(z)$ to be analytic in cartesian co-ordinates.

Proof:- Necessary condition:-

Let the function $f(z)$ be derivable for all values of z

i.e $f'(z)$ is exist $\forall z$

and $f(z)$ is analytic

$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \rightarrow (1)$

$\because z = x+iy$ then $\Delta z = \Delta x + i\Delta y$

$z + \Delta z = (x+\Delta x) + i(y+\Delta y)$

$\therefore f(z) = u(x,y) + i v(x,y)$

Now $f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y)$

from (1)

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y) - u(x,y) - i v(x,y)}{\Delta x + i\Delta y} \rightarrow (2)$$

Since the limit in (1) exists, so the limit value along any two directions must be equal.

Case (i):- If $\Delta z \to 0$ along x-axis.

In this case $\Delta z = \Delta x$

Now From (2) $f'(z) = \lim_{\Delta z \to 0} \left[\frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i \frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y} \right]$

$$f'(z) = \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right]$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \rightarrow (3)$$

$\therefore$ $f'(z)$ exists $\Rightarrow$ u_x, v_x are exists

Case (ii):- If $\Delta z \to 0$ along y-axis, $\Delta x = 0$

In this case $\Delta z = i\Delta y$

From (2).

$$f'(z) = \lim_{\Delta z \to 0} \left[\frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i \frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y} \right]$$

$$= \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \cancel{i} \frac{v(x, y+\Delta y) - v(x,y)}{\cancel{i}\Delta y} \right]$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \left(\frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = -i \right)$$

$$f'(z) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \rightarrow (4)$$

$\therefore$ $f'(z)$ exists $\Rightarrow$ u_y, v_y are exists

From (3) & (4)

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \qquad (\because f(z) \text{ exists})$$

Now equating real and imaginary parts

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore$ The condition is necessary

Sufficient condition:-

Let $f(z)$ be any function satisfying two conditions

i.e (i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Now we have to P.T $f(z)$ is derivable for all values of z

we know that

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$

Now expanding Taylor's Series in u, v and negleting second and higher order terms

i.e $f(z+\Delta z) = \left[u(x,y) + \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right)\right] + i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right)\right]$

by C-R equations

$$= [u(x,y) + iv(x,y)] + \left[\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right] + i\left[\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right]$$

$$= f(z) + \left[\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right] + i\left[-\frac{\partial u}{\partial y}\Delta x + \frac{\partial u}{\partial x}\Delta y\right]$$

$$f(z+\Delta z) - f(z) = \frac{\partial u}{\partial x}[\Delta x + i\Delta y] - i\frac{\partial u}{\partial y}[\Delta x + i\Delta y]$$

$$= \left[\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right][\Delta x + i\Delta y]$$

$$f(z+\Delta z) - f(z) = \left[\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right]\Delta z$$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Now taking limit both sides $\Delta z \to 0$

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z\to 0}\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

case(i) continu $\rightarrow$ CR eq.
case(ii) CR eq $\not\rightarrow$ co

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$\therefore$ u_x, v_x exists $\Rightarrow$ $f'(z)$ is exists $\forall z$

$\therefore$ The condition is sufficient

Note:- (1) C-R conditions are necessary but not sufficient i.e if the function $f(z) = u+iv$ is analytic then it must satisfies the CR eq's But the converse is not true.

(2) C-R eq's are used to determine whether a complex function is analytic (or) not

(3) C-R condition are sufficient if the partial derivatives u_x, u_y, v_x, v_y are continuous

[f is analytic $\Rightarrow$ C.R condition
analytic $\Leftarrow$ C.R condition + continuous PD]

Cauchy's - Riemann equations in polar form:-

Statement:- If $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ and $f(z)$ is derivable at $z_0 = r_0 e^{i\theta_0}$ then

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ; \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}}$$

Proof:- Let $z = re^{i\theta}$ then $f(z) = u(r,\theta) + iv(r,\theta)$

$$f(re^{i\theta}) = u + iv \rightarrow (1)$$

Now differentiating eq (1) w.r.to r partially

$$f'(re^{i\theta}) \cdot e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \longrightarrow (2)$$

Now differentiating eq (1) w.r.to 'θ' partially

$$f'(re^{i\theta}) \cdot re^{i\theta} \cdot i = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$f'(re^{i\theta}) e^{i\theta} = \frac{1}{ri}\left[\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right] \rightarrow (3)$$

from (2) & (3)

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}$$

compare both sides real and imaginary parts

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} &= \frac{-1}{r}\frac{\partial u}{\partial \theta} \end{aligned}\right\}$$ these are called C-R equations in polar form.

Corollary:- If $f'(z)$ exists,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

and $$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

Harmonic function:-

An Analytic function satisfying the laplace equation is called harmonic function

i.e $\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}=0 \Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)A=0 \Rightarrow \nabla^2 A=0$ is called Laplacian equation.

$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is called Laplacian operator.

Conjugate Harmonic Function:-

If $f(z)=u+iv$, is analytic function and u,v satisfies Laplace equation then If 'u' is harmonic where v is called conjugate harmonic.

Theorem:- prove that the real and imaginary parts of an analytic function $f(z)$ are harmonic

Proof:- Given $f(z)=u+iv \rightarrow (1)$ is analytic

Now we have to P.T the real and imaginary parts of $f(z)$. i.e u,v are harmonic

For this we have to P.T

$$\boxed{\nabla^2 u=0 \;;\; \nabla^2 v=0}$$

By Cauchy - Riemann equation

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \rightarrow (1)$ x y

$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \rightarrow (2)$ y x

Differentiating eq(1) w.r.to 'x' and eq(2) w.r.to 'y' partially

then $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y} \rightarrow (3)$

$\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x} \rightarrow (4)$

Adding (3) & (4)

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

$\Rightarrow \nabla^2 u=0$

$\therefore$ u is Harmonic

Now differentiating eq(1) w.r.to 'y' and eq(2) w.r.to 'x' partially

then $\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 v}{\partial y^2} \rightarrow (5)$

$\frac{\partial^2 u}{\partial x\partial y}=-\frac{\partial^2 v}{\partial x^2} \rightarrow (6)$

Now adding (5) & (6)

$\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}=0$

$\Rightarrow \nabla^2 v=0$

$\therefore$ V is harmonic

① prove the function $f(z)$ defined by

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \quad z \neq 0$$

$$= 0 \quad , z = 0$$ is continuous and the C-R equations are satisfied at the origin but $f'(0)$ does not exists.

Sol:- Given $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$

$$f(z) = \left(\frac{x^3-y^3}{x^2+y^2}\right) + i\left(\frac{x^3+y^3}{x^2+y^2}\right)$$

where $u = \frac{x^3-y^3}{x^2+y^2}$ & $v = \frac{x^3+y^3}{x^2+y^2}$

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$\left[u(x,0) = \frac{x^3 - 0}{x^2+0} = \frac{x^3}{x^2} = x \right.$$
$$\left. u(0,0) = 0 \right]$$

$$= \lim_{x\to 0} \frac{x}{x} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{-y}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{x}{x} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y}$$

$$= \frac{y}{y} = 1$$

$\therefore$ we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ C-R equations are satisfied at the origin

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{f(z)}{z}$$

$$= \lim_{z\to 0} \frac{(x^3-y^3) + i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

let $z \to 0$ along the path $y = mx$, then

$$f'(0) = \frac{(1-m^3) + i(1+m^3)}{(1+m^2)(1+im)}$$

which depends on m and hence is not unique.

Thus $f'(z)$ does not exist at $(0,0)$

(2) Show that $f(z) = xy + iy$ is every where continuous but is not analytic.

Sol:- Given $f(z) = xy + iy$

$f(z_0) = x_0y_0 + iy_0$ at the point z_0

WKT $z_0 = x_0 + iy_0$

$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} (xy + iy)$

$= \lim_{\substack{x \to x_0 \\ y \to y_0}} (xy + iy)$

$= x_0y_0 + iy_0$

$= f(z_0)$

$\lim_{\substack{y \to y_0 \\ x \to x_0}} (xy + iy) = \lim_{y \to y_0} x_0y + iy_0 = \lim_{y \to y_0} x_0y + iy$

$= x_0y_0 + iy_0$

$= f(z_0)$

$\therefore$ f is continuous every where $f(z) = u + iv$
$= xy + iy$

So $u = xy$, $v = y$

Then $u_x = y$ | $v_x = 0$
$u_y = x$ | $v_y = 1$

$\therefore u_x = v_y \Rightarrow y \neq 1$

$u_y = -v_x \Rightarrow x \neq 0$

Since C-R eqution conditions are not satisfied

$\therefore$ f is not analytic

(3) Show that $f(z) = z + 2\bar{z}$ is not analytic any where in the complex plane.

Sol:- Given $f(z) = z + 2\bar{z}$

$= (x+iy) + 2(x-iy)$

$= 3x - iy$

$f(z) = u + iv$

where $u = 3x$, $v = -y$

$u_x = 3$ | $v_x = 0$
$u_y = 0$ | $v_y = -1$

$\therefore u_x = v_y \Rightarrow 3 \neq -1$

$u_y = -v_x \Rightarrow 0 = 0$

$\therefore f(z)$ is not analytic anywhere since C-R conditions are not satisfied for any z.

(4) Show that $f(z) = \begin{cases} \dfrac{xy^2(x+iy)}{x^2+y^4}; & z \neq 0 \\ 0; & z=0 \end{cases}$

is not analytic at $z=0$ although C-R equations are satisfied at the origin.

Sol:-

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{f(z)-0}{z-0} = \lim_{z\to 0} \frac{f(z)}{z}$$

$$= \lim_{z\to 0} \frac{xy^2(x+iy)}{(x^2+y^4)(x+iy)}$$

$$= \lim_{z\to 0} \frac{xy^2}{x^2+y^4}$$

clearly $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{0}{x^2+0} = 0$

and $\displaystyle\lim_{\substack{y\to 0\\ x\to 0}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{0}{0+y^4} = 0$

$\therefore \displaystyle\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^4} = \lim_{\substack{y\to 0\\ x\to 0}} \frac{xy^2}{x^2+y^4} = 0$

Along the path $y=mx$,

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{x\to 0} \frac{x(m^2 \cdot x^2)}{x^2+m^4 \cdot x^4} = \lim_{x\to 0} \frac{m^2 x}{1+x^2 m^4} = 0$$

Also along the path $x=my^2$

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{y\to 0} \frac{(my^2)y^2}{m^2y^4+y^4} = \lim_{y\to 0} \frac{m}{m^2+1} \neq 0$$

limit value depends on m i.e on the path of approach and is different for the different paths followed and therefore limit does not exist. Hence $f(z)$ is not differentiable at $z=0$. Thus $f(z)$ is not analytic at $z=0$

To prove that C-R conditions are satisfied at the origin.

Let $f(z) = u+iv = \dfrac{xy^2(x+iy)}{x^2+y^4}$

Then $u(x,y) = \dfrac{x^2y^2}{x^2+y^4}$ & $v(x,y) = \dfrac{xy^3}{x^2+y^4}$ for $z \neq 0$

Also $u(0,0)=0$ & $v(0,0)=0$ $\quad [\because f(z)=0 \text{ at } z=0]$

Now $\dfrac{\partial u}{\partial x} = \mathop{Lt}\limits_{x\to 0} \dfrac{u(x,0)-u(0,0)}{x} = \mathop{Lt}\limits_{x\to 0} \dfrac{0-0}{x} = 0$

$$\frac{\partial u}{\partial y} = \mathop{Lt}\limits_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \mathop{Lt}\limits_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \mathop{Lt}\limits_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \mathop{Lt}\limits_{x\to 0} \frac{0-0}{x} = 0$$

and $\dfrac{\partial v}{\partial y} = \mathop{Lt}\limits_{y\to 0} \dfrac{v(0,y)-v(0,0)}{y} = \mathop{Lt}\limits_{y\to 0} \dfrac{0-0}{y} = 0$

Thus C-R equations are satisfied at the origin

Hence $f(z)$ is not analytic at $z=0$ eventhough C-R equations are Satisfied at the origin.

(5) If $f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2}, & z\neq 0 \end{cases}$ prove that $\dfrac{f(z)-f(0)}{z} \to 0$ as $z\to 0$ along any radius vector but not as $z\to 0$ along the curve $y=ax^3$

(or)

Test for analyticity at the origin for $f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2} & \text{for } z\neq 0 \\ 0 & \text{for: } z=0 \end{cases}$

(6) Show that the function $f(z)=\sqrt{|xy|}$ is not analytic at the origin although Cauchy-Riemann equations are satisfied at that point

Sol:- Let $f(z) = u(x,y)+iv(x,y) = \sqrt{|xy|}$

where $u=\sqrt{|xy|}$ & $v=0$

Then at the origin

$$\frac{\partial u}{\partial x} = \mathop{Lt}\limits_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \mathop{Lt}\limits_{x\to 0}\frac{0-0}{x} = 0$$

Similarly $\dfrac{\partial u}{\partial y} = \mathop{Lt}\limits_{y\to 0} \dfrac{u(0,y)-u(0,0)}{y} = \mathop{Lt}\limits_{y\to 0}\dfrac{0-0}{y} = 0$

$$\frac{\partial v}{\partial x} = \mathop{Lt}\limits_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \mathop{Lt}\limits_{x\to 0}\frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \mathop{Lt}\limits_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = 0$$

Hence C-R equations are satisfied at the origin

$$f'(0) = \mathop{Lt}\limits_{z\to 0} \frac{f(z)-f(0)}{z} = \mathop{Lt}\limits_{z\to 0} \frac{\sqrt{|xy|}-0}{x+iy}$$

Now let $z\to 0$ along $y=mx$, we get

$$f'(0) = \mathop{Lt}\limits_{x\to 0} \frac{\sqrt{|mx^2|}}{x(1+im)} = \mathop{Lt}\limits_{x\to 0} \frac{\sqrt{|m|}}{1+im}$$

This limit depends upon m and hence is not unique.

$\therefore f'(0)$ does not exist.

(7) If $w = \log z$, find $\frac{dw}{dz}$ and determine where w is non-analytic

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Sol:- Let $w = u + iv$

We have $z = x + iy = re^{i\theta}$, where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$

$\frac{dw}{dz} = f'(z)$

$\therefore \log(x+iy) = \log(re^{i\theta}) = \log r + i\theta$

$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$

$\therefore w = u + iv = \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$

So that $u = \frac{1}{2}\log(x^2+y^2)$; $v = \tan^{-1}(y/x)$

$\therefore \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$, $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$, $\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$ & $\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}$

The C-R equations are satisfied. Also the partial derivatives are continuous except at (0,0).

$\therefore w = \log z$ is analytic everywhere except at $z = 0$

$w = u + iv$

$\therefore \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{z}\ (z \neq 0)$

(8) Prove that z^n (n is a +ve integer) is analytic and hence find its derivative.

Sol:- Let $f(z) = z^n$

We have $z = x + iy = re^{i\theta}$

$f(z) = z^n$

$f(re^{i\theta}) = (re^{i\theta})^n = r^n e^{in\theta}$

$= r^n(\cos n\theta + i\sin n\theta)$

$= r^n\cos n\theta + ir^n\sin n\theta$

$= u(r,\theta) + iv(r,\theta)$

$\therefore u = r^n\cos n\theta$; $v = r^n\sin n\theta$

$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta$; $\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$

$\frac{\partial u}{\partial \theta} = -nr^n\sin n\theta$; $\frac{\partial v}{\partial \theta} = nr^n\cos n\theta$

By C-R equation in polar co-ordinates

$\boxed{\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r}}$

$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}nr^n\cos n\theta$

$= nr^{n-1}\cos n\theta$

$\boxed{\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r}}$

$\boxed{\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}}$

$\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{1}{r}(-nr^n\sin n\theta)$

$= -nr^{n-1}\sin n\theta$

$\boxed{\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}}$

Since the C-R equations are satisfied

The partial derivatives are continuous, therefore $f(z)=z^n$ is analytic

Differentiating $f(z)=u(r,\theta)+iv(r,\theta)$ partially w.r.to 'r', we get

$$\frac{\partial f}{\partial r}=f'(z)\frac{\partial z}{\partial r}$$

$$=f'(z)\cdot e^{i\theta}$$

$$f'(z)=\frac{1}{e^{i\theta}}\frac{\partial f}{\partial r}$$

$$=e^{-i\theta}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)$$

$$=e^{-i\theta}\left(nr^{n-1}\cos n\theta+i\,nr^{n-1}\sin n\theta\right)$$

$$=e^{-i\theta}\,nr^{n-1}(\cos n\theta+i\sin n\theta)$$

$$=nr^{n-1}\cdot e^{-i\theta}\cdot e^{in\theta}$$

$$=nr^{n-1}e^{i(n-1)\theta}$$

$$=n\,(re^{i\theta})^{n-1}$$

$$=n\,z^{n-1}$$

(9) prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Real } f(z)|^2=2|f'(z)|^2$ when $w=f(z)$ is analytic

Sol:- Let $f(z)=u+iv$

Then $\text{Real } f(z)=u$

Now $f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$

$$=u_x+iv_x$$

$$=u_x-iu_y \quad \text{(using C-R eqns)}$$

and $|f'(z)|=\sqrt{u_x^2+u_y^2} \longrightarrow (1)$

$$|f'(z)|^2=u_x^2+u_y^2$$

$\frac{\partial}{\partial x}(u^2)=2u\frac{\partial u}{\partial x}$

$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(u^2)\right)=\frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right)$

$\frac{\partial^2}{\partial x^2}(u^2)=uv'+vu'$

we have $\frac{\partial}{\partial x}(u^2)=2u\frac{\partial u}{\partial x}$

and $\frac{\partial^2}{\partial x^2}(u^2)=2\left[\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2}\right] \rightarrow (2)$

Similarly $\frac{\partial^2}{\partial y^2}(u^2)=2\left[\left(\frac{\partial u}{\partial y}\right)^2+u\frac{\partial^2 u}{\partial y^2}\right] \rightarrow (3)$

(1)+(3) gives

$$\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+u\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]\right] \rightarrow (4)$$

Since u is real part of $f(z)$, therefore it satisfies the Laplace's eqn

Hence $\nabla^2 u=0$

i.e $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \rightarrow (5)$

from (4) & (5), we have

$$\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$=2(u_x^2+u_y^2) \quad (\text{from } (1))$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\text{Real } f(z)|^2=2\,|f'(z)|^2$$

Hence the result.

(10) Show that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|=0$, where $f(z)$ is an analytic function.

Sol:- Taking $x=\frac{z+\bar z}{2}$; $y=\frac{z-\bar z}{2i}=\frac{-i}{2}(z-\bar z)$, we have

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial z}\right)$$

$$=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$$

and $\frac{\partial}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$

$x=\frac{1}{2}(z+\bar z)$

$\frac{\partial x}{\partial z}=\frac{1}{2}\cdot 1+0=\frac{1}{2}$

$\frac{\partial y}{\partial z}=\frac{1}{2i}\cdot 1-0=\frac{1}{2i}$

$\frac{\partial x}{\partial \bar z}=\frac{1}{2}$, $\frac{\partial y}{\partial \bar z}=\frac{1}{2i}$

$$\therefore \frac{\partial^2}{\partial z\partial \bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\cdot\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial^2}{\partial z\partial \bar z}=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

Hence $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|=4\frac{\partial^2}{\partial z\partial \bar z}\left(\frac{1}{2}\log|f'(z)|^2\right)$ $[\because |z|^2=z\bar z]$

(11)

$$=2\frac{\partial^2}{\partial z\partial \bar z}\left[\log|f'(z)|^2\right]$$

$$=2\frac{\partial^2}{\partial z\partial \bar z}\left[\log\{f'(z)\cdot f'(\bar z)\}\right]$$

$$=2\frac{\partial^2}{\partial z\partial \bar z}\left[\log f'(z)+\log f'(\bar z)\right]$$

$\frac{\partial^2}{\partial z\partial \bar z}\frac{1}{f'(z)}$

$\frac{\partial}{\partial \bar z}$ $\frac{\partial}{\partial z}[0]+\frac{\partial}{\partial z}\left[\frac{f''(\bar z)}{f'(\bar z)}\right]$

$\times 0$

$$=2\left[\frac{\partial}{\partial \bar z}\frac{f''(z)}{f'(z)}+\frac{\partial}{\partial z}\frac{f''(\bar z)}{f'(\bar z)}\right]=2(0+0)=0$$

Since $f(z)$ is analytic, $f'(z)$ is analytic, $f'(\bar z)$ is also analytic and $\frac{\partial f'(z)}{\partial \bar z}=0$, $\frac{\partial f'(\bar z)}{\partial z}=0$

(11) (i) prove that $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\partial \bar{z}}$

(ii) If $f(z)$ is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$

Sol:- (i) Let $z=x+iy$

Then $\bar{z}=x-iy$

$\therefore x=\frac{z+\bar{z}}{2}$ and $y=\frac{z-\bar{z}}{2i}=\frac{-i}{2}(z-\bar{z})$

Now $\frac{\partial x}{\partial z}=\frac{1}{2}$, $\frac{\partial x}{\partial \bar{z}}=\frac{1}{2}$ & $\frac{\partial y}{\partial z}=\frac{-i}{2}$, $\frac{\partial y}{\partial \bar{z}}=\frac{i}{2}$

Let $f=f(x,y)$. Then $f=f(z,\bar{z})$

we know that

$$\frac{\partial f}{\partial z}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)$$

$$=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)f$$

Similarly $\frac{\partial f}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)$

$$\therefore \frac{\partial^2 f}{\partial z\partial \bar{z}}=\frac{\partial}{\partial z}\cdot\frac{\partial f}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\cdot\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

$$=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f$$

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\partial \bar{z}}$$

(ii) WKT $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\partial \bar{z}}$

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4\frac{\partial^2}{\partial z\partial \bar{z}}|f(z)|^2$$

$$=4\frac{\partial^2}{\partial z\partial \bar{z}}f(z)\overline{f(z)}$$

$$=4\frac{\partial}{\partial \bar{z}}f'(z)\cdot\overline{f(z)}$$

$$=4f'(z)\overline{f'(z)}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$$

(12) Show that both the real and imaginary parts of an analytic function satisfies Laplace's equation (are harmonic)

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Sol:- Let $f(z) = u + iv$ be an analytic function.

By C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow (1) \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow (2)$$

eq (1) Diff partially w.r.to 'x'   eq (2) Diff partially w.r.to 'y'

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \rightarrow (3) \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \rightarrow (4)$$

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$\therefore$ u satisfies Laplace's equation, hence, u is harmonic.

Again eq (1) Diff partially w.r.to 'y'   eq (2) Diff partially w.r.to 'x'

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \rightarrow (5) \qquad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \rightarrow (6)$$

(5) + (6) gives, $\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0$

$\therefore$ V satisfies Laplace's equation. hence, v is harmonic.

Thus, both u & v are harmonic functions.

(13) Every analytic function $f(z) = u + iv$ defines two families of curves $u(x,y) = k_1$ and $v(x,y) = k_2$ forming an orthogonal system.

(or)

If $w = f(z)$ is an analytic function, then prove that the family of curves defined by $u(x,y) = \text{constant}$ cuts orthogonally the family of curves $v(x,y) = \text{constant}$.

Sol:- consider the two families of curves

$$u(x,y) = k_1 \rightarrow (1) \quad ; \quad v(x,y) = k_2 \rightarrow (2)$$

Let $f(z) = u + iv$ be the analytic function.

$u(x,y) = k_1$

By Cauchy-Riemann eqns

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow (3) \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow (4)$$

Diff (1) w.r.to 'x' $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1$ say $\frac{dy}{dx} = -\frac{\partial u}{\partial x} \times \frac{\partial y}{\partial u}$

$\therefore \frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = m_1$ say

Diff ② w.r.to 'x' $\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{dy}{dx}=0$

$$\frac{dy}{dx}=-\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}=\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}=m_2 \quad \left[\because -\frac{\partial v}{\partial x}=\frac{\partial u}{\partial y};\ \frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}\right]$$

Now $m_1 \times m_2 = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\times\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}=-1$

$\therefore\ m_1 m_2 = -1$

Hence, the curves ① & ② cut each other orthogonally

i.e They form an orthogonal system, which proves the result.

The two families are said to be the orthogonal trajectories of one another.

(14) If (i) $f(z)=u+iv=\frac{1}{z}$ (ii) $f(z)=u+iv=z^3$ Show that the curves $u(x,y)=c_1$ and $v(x,y)=c_2$ intersect orthogonally

(15) Show that the curves $r^n=\alpha\sec n\theta$ & $r^n=\beta\operatorname{cosec} n\theta$ cut orthogonally.

Sol:- Given curves can be written as

$r^n\cos n\theta=\alpha$ & $r^n\sin n\theta=\beta$

i.e $u(r,\theta)=\alpha$ & $v(r,\theta)=\beta$

Now $u(r,\theta)+iv(r,\theta)=\alpha+i\beta$

$=r^n\cos n\theta+ir^n\sin n\theta$

$=r^n(\cos n\theta+i\sin n\theta)$

$=r^n e^{in\theta}$

$=(re^{i\theta})^n=z^n$

which is an analytic function.

We know that if $w=u+iv$ is an analytic function, the curves of the family $u(x,y)=c_1$ cut orthogonally the curves of the family $v(x,y)=c_2$. Hence the result.

(16) Determine p such that the function $f(z)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\frac{px}{y}\right)$ be an analytic function.

Sol:- Let $f(z)=u+iv$

$=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\frac{px}{y}\right)$

Then $u=\frac{1}{2}\log(x^2+y^2)$ & $v(x,y)=\tan^{-1}\left(\frac{px}{y}\right)$

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$\therefore u_x = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2}$

$u_y = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2y = \frac{y}{x^2+y^2}$

$V_x = \frac{1}{1+\left(\frac{px}{y}\right)^2}\left(\frac{p}{y}\right) = \frac{py}{y^2+p^2x^2}$

and $V_y = \frac{1}{1+\left(\frac{px}{y}\right)^2}\left(\frac{-px}{y^2}\right) = \frac{-px}{y^2+p^2x^2}$

Clearly $u_x = v_y$ if $p=-1$

and $u_y = -v_x$ if $p=-1$

Hence $f(z)$ is analytic when $p=-1$

(17) Show that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic

(18) Find K such that $f(x,y) = x^3+3kxy^2$ may be harmonic and find its conjugate.

Sol:- we have $f(x,y) = x^3+3kxy^2$

$\therefore \frac{\partial f}{\partial x} = 3x^2+3ky^2, \; \frac{\partial f}{\partial y} = 6kxy$

and $\frac{\partial^2 f}{\partial x^2} = 6x, \; \frac{\partial^2 f}{\partial y^2} = 6kx$

Since $f(x,y)$ is harmonic.

$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

i.e $6x + 6kx = 0 \Rightarrow x(k+1)=0$

$6x(k+1) = 0 \quad (\because x \neq 0)$ (or) $k=-1$

Hence $f(x,y) = x^3-3xy^2$

Let $g(x,y)$ be the conjugate of $f(x,y)$. Then

$\boxed{dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy}$

$dg = -\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy$ (using C-R eqns)

$dg = -6kxy\,dx + (3x^2+3ky^2)dy$

This is exact D.E. Integrating

$(k=-1)$

$\int dg = +\int 6xy\,dx + \int (3x^2-3y^2)dy + c$

y const free from x terms

$= 6y\left(\frac{x^2}{2}\right) - 3\left(\frac{y^3}{3}\right) + c = 3x^2y - y^3 + c$

$z = u+iv$

$z = f+ig$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$

Milne-Thomson's Method:-

Let $f(z) = u(x,y) + iv(x,y) \rightarrow (1)$

Since $z = x+iy$, $\bar{z} = x-iy$, we have

$\boxed{x = \frac{z+\bar{z}}{2}}$ and $\boxed{y = \frac{z-\bar{z}}{2i}}$

$\therefore f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \rightarrow (2)$

Now considering this relation as a formal identity in two independent variables z & $\bar{z}$ putting $\bar{z} = z$ in (2), we get

$\boxed{f(z) = u(z,0) + iv(z,0)} \rightarrow (3)$

$\therefore$ eq (3) is same as (1), if we replace x by z, and y by 0

Thus to express any function interms of z, x by z & y by 0

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$, by C-R eq's

Let $\frac{\partial u}{\partial x} = \phi_1(x,y)$ & $\frac{\partial u}{\partial y} = \phi_2(x,y)$

Then $\boxed{f'(z) = \phi_1(x,y) - i\phi_2(x,y)} \rightarrow (4)$

Now to express $f'(z)$ completely interms of z, we replace x by z & y by 0 in (4)

$\therefore f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

Hence $\boxed{f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + c_1}$

where c_1 is a complex constant

Similarly, if $v(x,y)$ is given, we can find u such that $u+iv$ is analytic.

Let us use milne-Thomson's method

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$ [by C-R eq's]

$= \psi_1(x,y) + i\psi_2(x,y)$

$= \psi_1(z,0) + i\psi_2(z,0)$

$\therefore f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]\,dz + c_2$

where c_2 is a complex constant

① If $u = e^x[(x^2-y^2)\cos y - 2xy\sin y]$ is real part of an analytic function, find the analytic function.

Sol:- Let $f(z) = u + iv$

where $u = e^x[(x^2-y^2)\cos y - 2xy\sin y]$

Then $\frac{\partial u}{\partial x} = e^x[(x^2-y^2)\cos y - 2xy\sin y] + e^x[2x\cos y - 2y\sin y]$

and $\frac{\partial u}{\partial y} = e^x[-2y\cos y + (x^2-y^2)(-\sin y) - 2x\sin y - 2xy\cos y]$

$\frac{d}{dx}(\cos y) = -\sin y$

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\frac{d}{dx}(\sin y) = \cos y$

$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ by using C-R eq's

$f'(z) = e^x[(x^2-y^2)\cos y - 2xy\sin y + 2x\cos y - 2y\sin y]$
$\quad - ie^x[-2y\cos y + (y^2-x^2)\sin y - 2x\sin y - 2xy\cos y]$

By Milne-Thomson's method, $f'(z)$ is expressed interms of z by replacing x by z, and y by 0

Hence $f'(z) = e^z(z^2+2z) - ie^z(0)$

$= e^z(z^2+2z)$

Intigrating by parts w.r.to 'z', we get

$f(z) = e^z(z^2-2z+2) + 2e^z(z-1)$

$= e^z(z^2-2z+2+2z-2)$

$\boxed{f(z) = e^z z^2 + C}$, where C is a complex constant.

② Find and analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$

③ Find the analytic function whose imaginary part is $e^x(x\sin y + y\cos y)$

④ Show that xy^2 cannot be real part of an analytic function

(8) If $f(z) = u+iv$ is an analytic function of z and if $u-v = e^x(\cos y - \sin y)$, find f(z) interms of z. 15 15

Sol:- Given $u-v = e^x(\cos y-\sin y) \longrightarrow (1)$

Diff partially w.r.to x&y

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y) \longrightarrow (2)$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = e^x(-\sin y - \cos y)$$

WKT By C-R eq's

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ \& \ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = e^x(\sin y + \cos y) \longrightarrow (3)$$

$(2)+(3) \Rightarrow 2\frac{\partial u}{\partial x} = 2e^x\cos y \Rightarrow \frac{\partial u}{\partial x} = e^x\cos y \longrightarrow (4)$

$(3)-(2) \Rightarrow 2\frac{\partial v}{\partial x} = 2e^x\sin y \Rightarrow \frac{\partial v}{\partial x} = e^x\sin y \longrightarrow (5)$

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= e^x\cos y + ie^x\sin y$$

$$= e^x(\cos y + i\sin y)$$

$$= e^x e^{iy}$$

$$f'(z) = e^{x+iy} = e^z$$

Integrating, $\int f'(z)\,dz = \int e^z dz + c$

$$f(z) = e^z + c$$

(5) Find the analytic function whose real part is

(i) $\frac{x}{x^2+y^2}$ (ii) $\frac{y}{x^2+y^2}$

(6) Sol:- Let $f(z) = u+iv$ where $u = \frac{x}{x^2+y^2}$

$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial x}{\partial x} - i\frac{\partial u}{\partial y}$ (By C-R eq's)

$= \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{2ixy}{(x^2+y^2)^2}$

By Milne's Thomson's method $f'(z)$ is expressed interms of z by replacing x by z, & y by 0.

Hence $f'(z) = \frac{-z^2}{z^4} = \frac{-1}{z^2}$

Integrating, $f(z) = -\int\frac{dz}{z^2} + c = \frac{1}{z} + c$ where c is a complex constant

(ii) Let $f(z) = u+iv$ when $u = \frac{y}{x^2+y^2}$

$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (using C-R eq's)

$= \frac{-2xy}{(x^2+y^2)^2} - \frac{i(x^2-y^2)}{(x^2+y^2)^2}$

By Milne-Thomson's method $f'(z)$ is expressed interms of z by replacing x by z and y by 0

Hence $f'(z) = \frac{i(z^2-0)}{(z^2+0)^2} = -\frac{i}{z^2}$

Integrating, $\int f'(z)\,dz = \int\frac{-i}{z^2}dz$

$\therefore f(z) = \frac{i}{z} + c$, when c is a complex constant.

Complex Integration

Line integral:- suppose $f(z)$ is a complex function in the region R and C is a smooth curve in R consider an interval (a,b) and $a < x_1 < x_2 < \cdots < x_n < b$ are points in (a,b)

$\Delta_r = x_r - x_{r-1}$ are chord vectors, then

$$\lim_{n \to \infty} \sum_{r=1}^{n} \Delta x_r = \int_a^b f(z)\,dz$$

where the summation tends to a limit and independent of the points choice. The limit exists if $f(z)$ is continuous along the path

Let C be any continuous curve joining a to b in the Z-plane. Divide 'c' into n parts by the point $z_1, z_2 \cdots z_{n-1}$

Let $\delta z_r = z_r - z_{r-1}$

(or) Line integral:- Any integral which is to be evaluated along a line is called line integral
It is denoted $\int_C f(z)dz$
Note:- If the curve is closed the line integral is $\oint_C f(z)dz$

Evalution of the Integrals:-

$$\int f(z)\,dz = \int (u+iv)(dx+idy)$$
$$= \int (udx - vdy) + i(udy + vdx)$$

where u & v are functions of 'x'

properties of Line integral

1. $\int_C [f(z) \pm g(z)]\,dz = \int_C f(z)dz \pm \int_C g(z)dz$

2. $\int_C [k\, f(z)]\,dz = k\int_C f(z)dz$

3. $\int_{z_1}^{z_2} f(z)\,dz = -\int_{z_2}^{z_1} f(z)\,dz$

4. $\int_{-z_1}^{z_1} f(z)\,dz = 0$ (if $f(z)$ is an odd function of z)

5. $\int_{-z_1}^{z_1} f(z)\,dz = 2\int_{0}^{z_1} f(z)\,dz$ (if $f(z)$ is an even function of z).

6. If $z_1 < z_3 < z_2$, then $\int_{z_1}^{z_2} f(z)\,dz = \int_{z_1}^{z_3} f(z)\,dz + \int_{z_3}^{z_2} f(z)\,dz$

(1) Evaluate $\int_{0}^{1+i} (x^2 - iy)\,dz$ along the path (i) $y = x$ (ii) $y = x^2$

(2) Integrate $f(z) = x^2 + ixy$ from $A(1,1)$ to $B(2,8)$ along

(i) The straight line AB

(ii) The curve $C: x = t,\ y = t^3$

Sol:- (i) Given $\int_C f(z)\,dz = \int_{(1,1)}^{(2,8)} (x^2 + ixy)(dx + i\,dy)$

Along AB: Equation of AB passing through $A(1,1)$ & $B(2,8)$

is

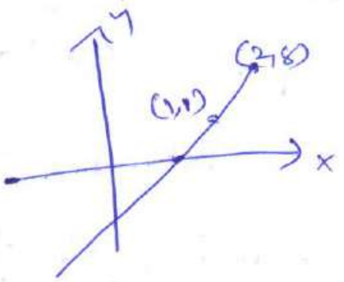
$$\frac{y-1}{8-1} = \frac{x-1}{2-1} \Rightarrow \frac{y-1}{7} = \frac{x-1}{1}$$

$$\Rightarrow y - 1 = 7x - 7$$

$$\Rightarrow 7x - 6 = y$$

$$\Rightarrow y = 7x - 6$$

$$\Rightarrow dy = 7\,dx$$



$$\therefore \int_{AB} f(z)\,dz = \int_{x=1}^{2} [x^2 + ix(7x-6)][dx + i7\,dx]$$

$$= (7i+1)\int_{x=1}^{2} [x^2 + 7ix^2 - 6ix]\,dx$$

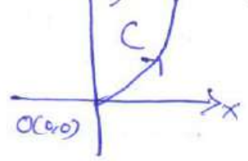
$$= (7i+1)\left[\frac{x^3}{3} + 7i\frac{x^3}{3} - 6i\frac{x^2}{2}\right]_1^2$$

$$= (1+7i)\left[\frac{8}{3} + 7i\cdot\frac{8}{3} - 6i\cdot 2 - \frac{1}{3} - 7i\cdot\frac{1}{3} + 6i\cdot\frac{1}{2}\right]$$

$$= \frac{(1+7i)}{3}[8 + 56i - 36i - 1 - 7i + 9i]$$

$$= \left[\frac{1+7i}{3}\right][22i + 7]$$

(ii) Along c whose parametric equations are

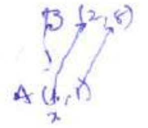


$x=t;\ y=t^3$

$dx=dt;\ dy=3t^2dt$

$A(1,1) \Rightarrow t=1\ ;\ B(2,8) \Rightarrow t=2$

$$\int_C f(z)\,dz = \int_{(1,1)}^{(2,8)} (x^2+ixy)(dx+idy)$$



$$= \int_{t=1}^{2} (t^2+it^4)(dt+i3t^2dt)$$

$$= \int_1^2 (t^2+it^4)(1+3it^2)\,dt$$

$$= \int_1^2 (t^2+it^4+3it^4-3t^6)\,dt$$

$$= \int_1^2 t^2+4it^4-3t^6\,dt$$

$$= \left[\frac{t^3}{3}+4i\frac{t^5}{5}-\frac{3t^7}{7}\right]_1^2 = \left[\frac{8}{3}+4i\cdot\frac{32}{5}-3\cdot\frac{128}{7}-\frac{1}{3}-\frac{4i}{5}+\frac{3}{7}\right]$$

$$= \frac{1}{45}[1441+1549]$$

(3) Evaluate $\int (2y+x^2)dx+(3x-y)dy$ along the parabola $x=2t,\ y=t^2+3$ joining (0,3) & (2,4).

$\int_{(0,3)}^{(2,4)}$

Sol:- At $(0,3) \Rightarrow t=0$

$x=2t,\ y=t^2+3$

$0=2t,\ 3=t^2+3$

$t=0,\ 0=t$

At $(2,4) \Rightarrow t=1$

$x=2t,\ 4=t^2+3$

$2=2t,\ t^2=4-3$

$t=1,\ t^2=1$

$t=1$

Substituting for x and y interms of t, we get

$$I = \int_{t=0}^{1} [2(t^2+3)+4t^2]\,2dt + \int_{t=0}^{1} [6t-t^2-3]\cdot 2t\,dt$$

$$= \left[\frac{24}{3}t^3+12t-\frac{2t^4}{4}-\frac{6t^2}{2}\right]_0^1$$

$$= 8+12-\frac{1}{2}-3 = \frac{33}{2}$$

Note:-

① A continuous arc without multiple points is called a Jordan arc In addition, if $x'(t)$ and $y'(t)$ are also continuous in the range $a \le t \le b$, then the arc is called a regular (or) smooth

② If $\int_C f(z)dz = \int_{t=a}^{b} (u+iv)(dx+idy)$

$$= \int_{t=a}^{b} (udx - vdy)dt + i\int_{t=a}^{b} (udy + vdx)dt$$

The integrands in the right hand side integrals are functions of t.

We get $\int_C f(z)dz$ by evaluating the two integrals on the right hand side

$\int_C f(z)dz$ is also called a contour integral

If the contour C is traced from t = b to a the value of the integral will be (−1) times the above integral.

④ Evaluate $\int_{1-i}^{2+i} (2x+1+iy)\,dz$ along $(1-i)$ to $(2+i)$

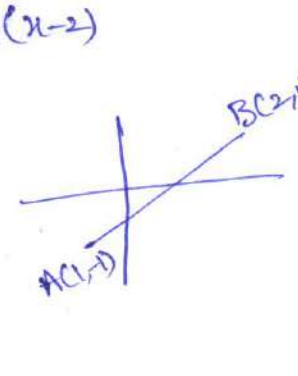
Sol:- Along $(1-i)$ to $(2+i)$ is the straight line AB joining $(1,-1)$ to $(2,1)$

The equation of AB is $y-1 = \left(\frac{-1-1}{1-2}\right)(x-2)$

$$y - 2x = -3$$

$$\Rightarrow y = 2x-3$$

$$\Rightarrow dy = 2dx$$



x varies from 1 to 2

$$\int_{1-i}^{2+i} (2x+1+iy)dz = \int_1^2 (2x+1)dx - (2x-3)(2dx) + i\left[(2x-3)dx + (2x+1)2dx\right]$$

$$= \int_1^2 (-2x+7)dx + i\,(6x-1)dx$$

$$= \left[-2\cdot\frac{x^2}{2} + 7x + i\left(6\frac{x^2}{2} - x\right)\right]_1^2 = 4+8i$$

⑤ Evaluate $\int_C \bar{z}\,dz$, $z=0$ to $4+2i$ Along c given by

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(a) t^2+it

(b) Along the line $z=0$ to 2 and there from 2 to $4+2i$

Sol:- (a) Given

$t^2+it = x+iy \Rightarrow x=t^2;\ y=t$, t varies from 0 to 2

$dx=2t\,dt;\ dy=dt$

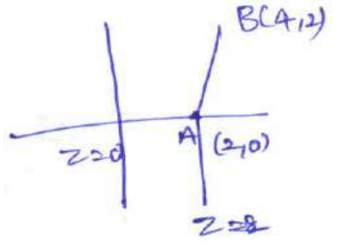
$$\int_C (x-iy)(dx+idy) = \int_C (x\,dx+y\,dy)+i(x\,dy-y\,dx)$$

$$=\int_0^2 (2t^3dt + t\,dt) + i(t^2dt - 2t^2dt)$$

$$=\left[\left(2\frac{t^4}{4}+\frac{t^2}{2}\right)+i\left(\frac{t^3}{3}-\frac{2t^3}{3}\right)\right]_0^2$$

$$=10-\frac{8i}{3}$$

(b) Along OA, $y=0 \Rightarrow dy=0$



x varies from 0 to 2

$$\int_{OA}\bar{z}\,dz = \int_0^2 x\,dx = 2$$

Along AB, the equation of AB

$$(y-2) = \frac{2-0}{4-2}(x-4)$$

(2,0) (4,2)
$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$$y-2 = (x-4)$$

$$y = x-2$$

$$\Rightarrow dy = dx$$

x varies from 2 to 4

$$\int_2^4 [x-i(x-2)][dx+idx] = \int_2^4 (x+x-2)dx + i(x-x+2)dx$$

$$=\int_2^4 (2x-2)dx + \int_2^4 2i\,dx$$

$$=\left[\frac{\cancel{2}x^2}{\cancel{2}}-2x\right]_2^4 + 2i[x]_2^4$$

$$= 8+4i$$

(6) Evaluate $\int_{z=0}^{z=1+i} [(x^2+2xy)+i(y^2-x)]\,dz$ along $y=x^2$

Sol:- Along $y=x^2 \Rightarrow dy = 2x\,dx \Rightarrow dy = 2\sqrt{y}\,dx$

x varies from 0 to 1

$$\int_0^{1+i} [(x^2+2xy)+i(y^2-x)]\,[dx+i\,dy] = \int_0^1 [(x^2+2x\cdot x^2)+i(x^2-x)]\,[dx+i\,2x\,dx]$$

$$= \int_0^1 [(x^2+2x^3)+i(x^2-x)]\,[1+2ix]\,dx$$

$$= \int_0^1 [x^2+2x^3+i(x^2-x)+2i(x^3+2x^4) - 2(x^3-x^2)]\,dx$$

$$= \int_0^1 [2x^2+i(-x+x^2+x^3+2x^4)]\,dx$$

$$= \left[\frac{2x^3}{3}+i\left(-\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{2x^5}{5}\right)\right]_0^1$$

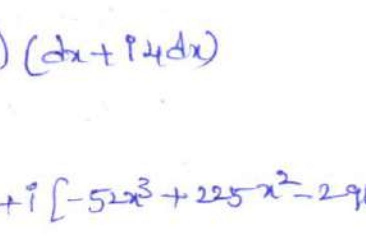
$$= \frac{2}{3}+i\left(-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{2}{5}\right)$$

$$= \frac{2}{3}+i\,($$

$$= \frac{7}{6}+i$$

(7) Evaluate $\int_{1-i}^{2+3i} (z^3+z)\,dz$ along the line joining $z=1-i$ to $2+3i$

Sol:- The line joining $A(1,-1)$ & $B(2,3)$ is



$$(y-3) = \left(\frac{3+1}{2-1}\right)(x-2)$$

$$y = 4x-5$$

$$\Rightarrow dy = 4dx$$

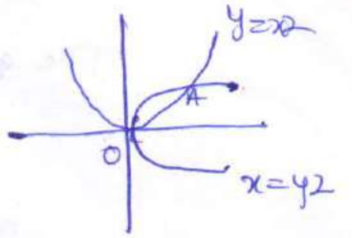
$$\int_{1-i}^{2+3i} (z^3+z)\,dz = \int_1^2 (x+i(4x-5))^3 + (x+i(4x-5))\,(dx+i4dx)$$

$$= (1+4i)\int_1^2 \{[-47x^3+120x^2-74x]+i[-52x^3+225x^2-296x+120]\}\,dx$$

$$= \frac{-125}{4} - 23i$$

(8) $\int_{(0,0)}^{(1,1)} (3x^2+4xy+ix^2)\,dz$ along $y=x^2$ Ans $\left(\frac{3}{2}+\frac{103i}{30}\right)$

(9) Evaluate $\int_C (y^2+2xy)dx + (x^2-2xy)dy$ where C is the boundary of the region by $y=x^2$ & $x=y^2$



Sol:- C_1: Along OA $y=x^2 \Rightarrow dy=2x\,dx$

x varies from 0 to 1

$$\int_{C_1} (y^2+2xy)dx + (x^2-2xy)dy = \int_0^1 (x^4+2x^3)dx + (x^2-2x^3)2x\,dx$$

$$= \frac{2}{5}$$

C_2: Along AO $x=y^2 \Rightarrow dx=2y\,dy$

y varies from 1 to 0

$$\int_{C_2} (y^2+2xy)dx + (x^2-2xy)dy = \int_1^0 (y^2+2y^3)2y\,dy + (y^4-2y^3)dy$$

$$= -1$$

$$\therefore \int_C (y^2+2xy)dx + (x^2-2xy)dy = -1+\frac{2}{5} = \frac{3}{5}$$

(10) prove that $\int_C \frac{dz}{z-a} = 2\pi i$ where C is $|z-a|=R$.

Sol:- The given curve C is $|z-a|=R$

$z-a = Re^{i\theta}$.

θ varies from 0 to 2π

R is the radius of the circle

$$dz = Rie^{i\theta}d\theta$$

$$\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{Rie^{i\theta}d\theta}{Re^{i\theta}}$$

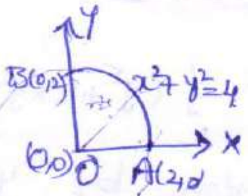
$$= i\int_0^{2\pi} d\theta = 2\pi i.$$

(11) Evaluate $\int_C (x-2y)dx + (y^2-x^2)dy$ where C is the boundary of the first quadrant of the circle $x^2+y^2=4$

Sol:- C is the boundary of the first quadrant of the circle

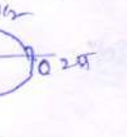
$x^2+y^2=4$

$\therefore \theta$ varies from 0 to $\frac{\pi}{2}$



$\theta \to 0$ to $\frac{\pi}{2}$
$r \to 0$ to 2

The parametric co-ordinates are



$x = 2\cos\theta$; $y = 2\sin\theta$

$dx = -2\sin\theta d\theta$; $dy = 2\cos\theta d\theta$

$$\int_C (x-2y)dx + (y^2-x^2)dy = \int_0^{\pi/2} (2\cos\theta - 4\sin\theta)(-2\sin\theta\, d\theta) + (4\sin^2\theta - 4\cos^2\theta).(2\cos\theta d\theta)$$

$$= \int_0^{\pi/2} [-4\cos\theta\sin\theta + 8\sin^2\theta + 8\sin^2\theta\cos\theta - 8\cos^3\theta]d\theta$$

$$= \int_0^{\pi/2} -2\sin 2\theta + 4(1-\cos 2\theta) + 8\sin^2\theta\cos\theta - \overset{2}{\cancel{8}}\cdot\frac{1}{\cancel{4}}[\cos 3\theta + 3\cos\theta]\,d\theta$$

$$= \int_0^{\pi/2} [-2\sin 2\theta + 4 - 4\cos 2\theta + 8\sin^2\theta\cos\theta - 2\cos 3\theta - 6\cos\theta]d\theta$$

$$= \left[\cancel{2}\frac{\cos 2\theta}{\cancel{2}} + 4\theta - 4\frac{\sin 2\theta}{2} + 8\frac{\sin^3\theta}{3} - 2\frac{\sin 3\theta}{3} - 6\sin\theta\right]_0^{\pi/2}$$

$$= \left\{2\frac{[-1]}{2} + 4\frac{\pi}{2} - 4\frac{[0]}{2} + 8\frac{[1]^3}{3} - 2\frac{[-1]}{3} - 6[1]\right\} - \left\{2\frac{(1)}{2} + 0 - 0 + 0 - 0 - 0\right\}$$

$$= -1 + 2\pi - 0 + \frac{8}{3} + \frac{2}{3} - 6 - 1$$

$$= 2\pi + \frac{10}{3} - 8$$

$$= 2\pi + \frac{10-24}{3}$$

$$= 2\pi - \frac{14}{3}$$

Cauchy's integral Formula:-

Statement:- Let $f(z)$ be an analytic function everywhere on and within a closed contour c. If $z=a$ is any point with in 'C', then 20

$$f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz$$

Proof:- consider the function $\frac{f(z)}{z-a}$ which is analytic at all points except at $z=a$ draw a small circle C_1 with centre 'a' and radius 'r'

Now the function $\frac{f(z)}{(z-a)}$ is analytic b/w C & C_1

Extension of cauchy's theorem

$$\int_C \frac{f(z)}{z-a}dz=\int_{C_1}\frac{f(z)}{z-a}dz \rightarrow (1)$$

for any point on C_1, $z-a=re^{i\theta}$

$dz=ire^{i\theta}d\theta$

θ varies from 0 to 2π

$$\int_C \frac{f(z)}{z-a}dz=\int_0^{2\pi}\frac{f(a+re^{i\theta})}{re^{i\theta}}\cdot ire^{i\theta}d\theta$$

$$=i\int_0^{2\pi}f(a+re^{i\theta})d\theta \rightarrow (2)$$

In the limiting, the circle c. shrunks to a point a

i.e $r\to 0$, the integral (2) becomes

$$\int_C \frac{f(z)}{z-a}dz=if(a)\int_0^{2\pi}d\theta$$

$$=if(a)\left[\theta\right]_0^{2\pi}$$

$$=2\pi i\, f(a) \quad (\text{by } (1))$$

$$\Rightarrow f(a)=\frac{1}{2\pi i}\int_C\frac{f(z)}{z-a}dz$$

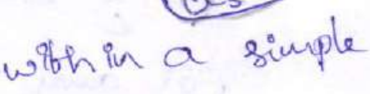
Simple curve:- A curve which does not intersect with itself.
A simple curve which is closed is called simple closed curve. 20

Multiple curve:- A multiple curve crosses itself.

Connected region:- A region is said to be connected region if any two points of the region D can be connected by a curve lies entirely with in the region [initial point = end point].

Simply connected region:- A simply connected region is a region such that every closed curve lying in it can be contracted indefinitely with out passing out of it.

Multiply connected region:- A region which is bounded by more than two curves is called a multiple connected region.



Cauchy's Integral theorem:-

Let $f(z)$ be an analytic on and with in a simple closed contour C and let $f'(z)$ be continuous there.

Then $\int_C f(z)\,dz = 0$

$\left(\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy\right)$

Proof:- we have $f(z) = u(x,y) + iv(x,y)$

and $z = x+iy$

$\Rightarrow dz = dx + i\,dy$

$\therefore \int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \rightarrow (1)$

$M=u,\ N=-v$ $\qquad$ $M=v,\ N=u$

Since $f'(z)$ is continuous therefore $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region R enclosing by curve C.

Hence, appling Green's theorem, (1) becomes

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \rightarrow (2)$$

Since $f(z)$ is analytic, u, v satisfies C-R eqs $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Substituting these in (2), we get

$$\boxed{\int_C f(z)\,dz = 0}$$

Generalization of Cauchy's integral formula:-

If $f(z)$ is analytic on and with in a simple closed curve C and if 'a' is any point with in C, (21)

Then $f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$

Proof:- We have $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz$$

$$f'(a) = \lim_{\Delta a\to 0}\frac{f(a+\Delta a)-f(a)}{\Delta a}$$

$$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-(a+\Delta a)} - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

$$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}\frac{1}{2\pi i}\int_C f(z)\left[\frac{1}{z-(a+\Delta a)}-\frac{1}{z-a}\right]dz$$

$$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}\cdot\frac{1}{2\pi i}\left[\int_C f(z)\cdot\frac{\Delta a\,dz}{(z-a-\Delta a)(z-a)}\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$$

Using principle of Mathmatical induction, we can prove that

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

$$\oint_C \frac{\frac{z-1}{(z+1)^2}}{(z-2)}$$

$$\int_C \frac{f(z)}{(z-a)}dz = 2\pi i f(a)$$

(1) Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$, where C is $|z-i|=2$

Sol:- The singularities of $\frac{z-1}{(z+1)^2(z-2)}$ are given by

$\sqrt{2}=2$
$2\neq 4$
$2-4=0$
$-2\neq 0$

$$(z+1)^2(z-2)=0 \Rightarrow z=-1 \,\&\, z=2$$

(or)

$z=-1$ lies inside the circle since $|-1-i|-2<0$ $\rightarrow$ $\begin{cases}\sqrt{1^2+1^2}-2<0\\ \sqrt{2}-2<0\\ 1.414-2<0\\ -0.6<0\end{cases}$

$z=2$ lies out-side the circle since $|2-i|-2>0$ $\rightarrow$ $\begin{cases}\sqrt{4+1}-2>0\\ \sqrt{5}-2>0\\ 2.14-2>0\\ 0.14>0\end{cases}$

(or)

$\sqrt{5}=2$
$5=4$
$5-4\neq 0$
$1\neq 0$

The given integral can be written as

$$\int_C \frac{z-1}{(z+1)^2(z-2)}dz = \int_C \frac{\left\{\frac{(z-1)}{(z-2)}\right\}}{(z+1)^2}dz \rightarrow (1)$$

The derivative of Analytic function is given by

$$\int_C \frac{f(z)}{(z-a)^{n+1}}dz = 2\pi i\frac{f^n(a)}{n!} \rightarrow (2)$$

From (1) & (2)

$$f(z) = \frac{z-1}{z-2}, \quad a=-1, \; n=1$$

$$\therefore f'(z) = \frac{d}{dz}\left(\frac{z-1}{z-2}\right)$$

$$= \frac{(z-2)\cdot 1 - (z-1)\cdot 1}{(z-2)^2}$$

$$= \frac{z-2-z+1}{(z-2)^2} = \frac{-1}{(z-2)^2}$$

$n=1$

$$f'(-1) = -\frac{1}{9}$$

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{1+1}}dz$$

Substituting in (2), we get

$$\int_C \frac{(z-1)}{(z+1)^2(z-2)}dz = \frac{2\pi i\left(-\frac{1}{9}\right)}{1!}$$

$$= -\frac{2}{9}\pi i$$

(2) Determine $\int_C \frac{z+4}{z^2+2z+5}dz$ where C is the circle

(i) $|z|=1$ (ii) $|z+1-i|=2$ (iii) $|z+1+i|=2$

Evaluate $\int_C \frac{e^z}{z(1-z)^3}dz$

If
(i) 0 lies inside C & 1 lies outside
(ii) 1 lies inside C & 0 lies outside 'C'
(iii) both lie inside C
where C is
(i) $|z|=\frac{1}{2}$, (ii) $|z-1|=\frac{1}{2}$
(iii) $|z|=2$

Sol:-
(i) when C is the circle $|z|=1$

i.e $x^2+y^2=1$

$x^2+y^2-1=0 \Rightarrow f(x,y)=0$

Singularities of $\frac{z+4}{z^2+2z+5}$ are given by

i.e. $z^2+2z+5=0$

$(z+1)^2+4=0$

$(z+1)^2-(2i)^2=0$

$(z+1-2i)(z+1+2i)=0$

$z=-1+2i, \; z=-1-2i$

$$\int_C \frac{\frac{z+4}{(z-b)}}{(z-a)}dz \qquad \int_C \frac{z+4}{(z+1+2i)(z+1-2i)}dz$$

Both $(-1,2)$ & $(-1,-2)$ lie out side the circle because of substituting the values of x & y in that eqn. of the circle

$f(x,y)>0$ in both the cases

$\therefore f(z) = \frac{z+4}{z^2+2z+5}$ is analytic at all points with in and on the circle $|z|=1$. $\therefore$ By cauchy's integral theorem $\int_C \frac{z+4}{z^2+2z+5}dz=0$

(ii) when C is the circle $|z+1-i|=2$

$$\Rightarrow (x+1)^2+(y-1)^2=4$$

The point $(-1,-2)$ i.e $z=-1-2i$ does not lie within 'C' where as the point $(-1,2) \Rightarrow$ i.e $z=-1+2i$ lies inside.

$$\therefore \int_C \frac{z+4}{(z+1+2i)(z+1-2i)}\,dz = \int_C \frac{\frac{z+4}{(z+1+2i)}}{(z+1-2i)}\,dz$$

$$= \int_C \frac{f(z)}{z-a}\,dz \qquad f(z) = \frac{z+4}{z+1+2i} \ \& \ a=-1+2i$$

By cauchy's integral formula

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \longrightarrow (1)$$

$$\text{But } f(z) = \frac{z+4}{z+1+2i}$$

$$f(a) = f(-1+2i) = \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$$

Substituting in (1), we get

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = 2\pi i\left(\frac{3+2i}{4i}\right) = \frac{\pi}{2}(3+2i)$$

(iii) where C is the circle $|z+1+i|=2 \Rightarrow (x+1)^2+(y+1)^2=4$

The point $(-1,-2)$ i.e $z=-1-2i$ lies inside where as the point $z=-1+2i$ lies out side the circle

$$\therefore \int_C \frac{z+4}{(z+1+2i)(z+1-2i)}\,dz = \int_C \frac{\frac{z+4}{z+1-2i}}{z+1+2i}\,dz$$

$$= \int_C \frac{f(z)}{(z-a)}\,dz$$

$$\text{where } f(z) = \frac{z+4}{z+1-2i} \ \& \ a=-1-2i$$

$$\text{Hence } \int_C \frac{z+4}{(z+1+2i)(z+1-2i)}\,dz = 2\pi i\, f(a)$$

$$= 2\pi i\, f(-1-2i)$$

$$= 2\pi i\left(\frac{-1-2i+4}{-1-2i-2i+1}\right)$$

$$= 2\pi i\left(\frac{3-2i}{-4i}\right) = \frac{\pi}{2}(2i-3)$$

③ Use Cauchy's integral formula to evaluate $\int_C \frac{e^z}{(z^2+\pi^2)^2}dz$ where C is the circle $|z|=4$

Sol:- Given

$$\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}$$

$f(z)=e^z$ is analytic with in the circle $|z|=4$ and the two singular points $z=\pm\pi i$ lies inside 'C'

Let $$\frac{1}{(z^2+\pi^2)^2} = \frac{1}{(z+\pi i)^2(z-\pi i)^2}$$

$$= \frac{A}{z+\pi i}+\frac{B}{(z+\pi i)^2}+\frac{C}{z-\pi i}+\frac{D}{(z-\pi i)^2}$$

solving for A, B, C & D, we get

$$A=\frac{7}{2\pi^3 i},\ B=\frac{-1}{4\pi^2},\ C=\frac{-7}{2\pi^3 i},\ \&\ D=\frac{-1}{4\pi^2}$$

$$\therefore \int_C \frac{e^z}{(z^2+\pi^2)^2}dz = \frac{7}{2\pi^3 i}\int_C \frac{e^z}{(z+\pi i)}dz - \frac{1}{4\pi^2}\int_C \frac{e^z}{(z+\pi i)^2}dz - \frac{7}{2\pi^3 i}\int_C \frac{e^z}{z-\pi i}dz - \frac{1}{4\pi^2}\int_C \frac{e^z}{(z-\pi i)^2}dz$$

$\therefore$ By Cauchy's integral formula.

$$\int_C \frac{e^z}{(z^2+\pi^2)^2}dz = \frac{7}{2\pi^3 i}2\pi i\, f(-\pi i) - \frac{1}{4\pi^2}2\pi i\, f'(-\pi i) - \frac{7}{2\pi^3 i}2\pi i\, f(\pi i) - \frac{1}{4\pi^2}2\pi i\, f(\pi i)$$

$$= \frac{7}{\pi^2}e^{-i\pi} - \frac{i}{2\pi}e^{-i\pi} - \frac{7}{\pi^2}e^{i\pi} - \frac{i}{2\pi}e^{i\pi} = \frac{i}{\pi}$$

④ Evaluate $\int_C \frac{z^3-\sin 3z}{(z-\frac{\pi}{2})^3}dz$ with $C: |z|=2$ using Cauchy's integral formula.

Sol:- Given $f(z)=z^3-\sin 3z$ is analytic inside the circle $C: |z|=2$ and the singular point $a=\frac{\pi}{2}$ lie inside C

$\therefore$ By Cauchy's integral formula.

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}dz, \text{ we get}$$

$$\int_C \frac{z^3-\sin 3z}{(z-\frac{\pi}{2})^3}dz = \pi i f''(\tfrac{\pi}{2})$$ 23

$$= \pi i \frac{d^2}{dz^2}[z^3-\sin 3z]_{z=\frac{\pi}{2}}$$

$$= \pi i \frac{d}{dz}(3z^2-3\cos 3z)_{z=\frac{\pi}{2}}$$

$$= \pi i (6z+9\sin 3z)_{z=\frac{\pi}{2}}$$

$$= \pi i \left[6\cdot\frac{\pi}{2}+9\sin\frac{3\pi}{2}\right]$$

$$= (3\pi-9)\pi i = 3\pi i(\pi-3)$$

(5) Evaluate $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz$, where C is the circle $|z|=3$ using integral formula.

Sol:- $f(z) = \sin\pi z^2+\cos\pi z^2$ is analytic within the circle $|z|=3$. and the singular points $a=1,2$ lie inside C.

$$\therefore \int_C \frac{f(z)}{(z-1)(z-2)}dz = \int_C \left[\frac{1}{z-2}-\frac{1}{z-1}\right]f(z)dz$$

$$= \int_C \frac{f(z)}{z-2}dz - \int_C \frac{f(z)}{z-1}dz$$

$$= 2\pi i f(2) - 2\pi i f(1)$$ (using cauchy's integral formula)

$$= 2\pi i[\sin 4\pi+\cos 4\pi - \sin\pi - \cos\pi]$$

$$= 2\pi i[1-(-1)] = 4\pi i$$

(6) Evaluate $\int_C \frac{\log z}{(z-1)^3}dz$ where $C: |z-1|=\frac{1}{2}$ using Cauchy's integral formula.

Sol:- since $f(z)=\log z$ is analytic with in C and the singular point $a=1$ lies inside C

$\therefore$ By cauchy's integral formula.

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

we get $f''(a) = \frac{2!}{2\pi i}\int_C \frac{\log z}{(z-1)^3}dz$

$$\therefore f''(1) = \frac{2!}{2\pi i}\int \frac{\log z}{(z-1)^3}dz$$

$$\int_C \frac{\log z}{(z-1)^3}dz = \pi i f''(1)$$

$$= \pi i\left(\frac{-1}{z^2}\right)_{z=1}$$

$$= -\pi i$$

(7) Evaluate $\int_C \frac{z^3+z^2+2z-1}{(z-1)^3}\,dz$ where C is $|z|=3$ using cauchy's integral formula.

(8) using cauchy's integral formula, evaluate $\int_C \frac{z^4}{(z+1)(z-i)^2}\,dz$ where C is the ellipse $9x^2+4y^2=36$

(9) Find $f(2)$ and $f(3)$ if $f(a) = \int_C \frac{(2z^2-z-2)dz}{z-a}$ where C is the circle $|z|=2.5$ using cauchy's integral formula.

Sol:- Given $f(a) = \int_C \frac{2z^2-z-2}{z-a}\,dz$

(i) $a=2$ lies inside the circle $C: |z|=2.5$

let $\phi(z) = 2z^2-z-2$

By Cauchy's integral formula

$$\phi(a) = \frac{1}{2\pi i}\int_C \frac{\phi(z)}{(z-a)}\,dz$$

$$\Rightarrow 2\pi i\,\phi(a) = \int_C \frac{\phi(z)}{z-a}\,dz = f(a)$$

$$\Rightarrow f(a) = 2\pi i\,\phi(a)$$
$$= 2\pi i(2a^2-a-2)$$

$z-a=0$ $z=2$

$|z|=2.5$

(i) $f(2) = 2\pi i(8-2-2) = 8\pi i$

(ii) Taking $a=3$, we get

$$f(3) = \int_C \frac{2z^2-z-2}{z-3}\,dz$$

Now the point $z=3$ lies outside C

Hence the integrand is analytic within and on C

$\therefore$ By cauchy's theorem,

$$f(3) = \int_C \frac{2z^2-z-2}{z-3}\,dz = 0$$

UNIT-IV

[COMPLEX POWER SERIES & CONTOUR INTEGRATION]

Introduction: In this unit, we shall discuss the methods of expanding a given function about a point a in powers of $z-a$. as we proceed, we recognize that this theory enables us in evaluating certain real & complex integrals easily. Here we discuss Taylors series & Laurent series expansion of $f(z)$ about pt a.

In this unit we also discuss about Residue theorem which is useful to evaluate certain real integrals.

Basic Definitions: * **Sequence:** A sequence $\{z_n\}$ is a function from $N \to \mathbb{C}$. ie $z_n : N \to \mathbb{C}$ & it is denoted by $\{z_n\}$.

* A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers is said to be cgt to z_0 if for each $\epsilon > 0$ $\exists$ a +ve integer $N \ni$ $|z_n - z_0| < \epsilon$, $\forall$ $n \geq N$. ie $z_n \in N_\epsilon(z_0)$ $\forall$ $n \geq N$.

then we write $\lim_{n\to\infty} z_n = z_0$.

* A sequence is said to be dgt if it not cgt

* If $\lim_{n\to\infty} z_n = z_0$ then 1) $\lim_{n\to\infty} |z_n| = |z_0|$ 2) the seq $\{z_n\}_{n=1}^{\infty}$ is bdd

Series: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, the n^{th} partial sum of the sequence is called series & it is denoted by $\sum_{n=1}^{\infty} a_k$.

Series: let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers, the series $\sum_{k=1}^{\infty} a_k$ is said to be converges, if $\{s_n\}_{n=1}^{\infty}$, the sequence of n^{th} partial sum cgs to s, then we write $\lim_{n\to\infty} s_n = s$.

* A series is said to be dgt if it is not cgt.
* A series $\sum_{k=1}^{\infty} a_k$ of complex numbers is said to be absolutely cgt if $\sum_{k=1}^{\infty} |a_k|$ is cgt.
* Abs cgs $\Rightarrow$ cgt but converse is not true.

Power series: Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex no's

The series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is called a power series of z_0.

* The series $\sum_{n=0}^{\infty} a_n z^n$ is a power series about the origin.

* If a series $\sum_{K=0}^{\infty} a_K$ cgs at every point of circle 'C' & dgs at every point outside the circle 'C', then such a circle 'c' is said to be the circle of convergence of the series $\sum_{K=0}^{\infty} a_k$. The radius R of the circle 'C' is called the radius of convergence of the series $\sum_{K=0}^{\infty} a_k$.

* The formula to find Radius of convergence (R) is $\frac{1}{R} = \underset{n\to\infty}{\text{lt}} \sup \left|\frac{a_{n+1}}{a_n}\right|$

(OR) $\frac{1}{R} = \underset{n\to\infty}{\text{lt}} \sup |a_n|^{1/n}$

prob(1) Find the circle of convergence of the series $\sum_{n=1}^{\infty} (\log z)^n \cdot z^n$

Sol

we have $\sum_{n=1}^{\infty} (\log z)^n z^n = \sum_{n=1}^{\infty} a_n z^n$

on comparing

$a_n = (\log z)^n$

W.K.T $\frac{1}{R} = \underset{n\to\infty}{\text{lt}} \sup |a_n|^{1/n}$

$= \underset{n\to\infty}{\text{lt}} \sup |(\log z)^n|^{1/n}$

$= \underset{n\to\infty}{\text{lt}} = \infty$

$\Rightarrow \frac{1}{R} = \infty$

$\Rightarrow R = 0$

Radius of convergence $= 0$

i.e circle with zero radius.

hence the circle of convergence is $|z| = 0$

(2) Find the circle of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$

Sol we have $a_n = \frac{(-1)^{n-1}}{(2n-1)!}$ $a_{n+1} = \frac{(-1)^n}{(2n+1)!}$

$\frac{1}{R} = \underset{n\to\infty}{\text{lt}} \sup \left|\frac{a_{n+1}}{a_n}\right| = \underset{n\to\infty}{\text{lt}} \sup \left|\frac{(2n-1)!}{(2n+1)(2n)(2n-1)!}\right| = \underset{n\to\infty}{\text{lt}} \left|\frac{1}{2n(2n+1)}\right| = 0 \Rightarrow \frac{1}{R} = 0 \Rightarrow R = \infty$

$\therefore$ The given series cgs everywhere in complex plane i.e circle with ∞ radius

Taylor's theorem: Let $f(z)$ be analytic at all points (2)
within a circle C with centre at 'a' & radius r. then at each point 'z' within 'c'

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{(z-a)^n}{n!} f^n(a) + \cdots \quad (1)$$

ie the series on the right hand side in (1) converges to $f(z)$ whenever $|z-a| < r$.

* The Expansion in (1) on the R.H.S is called the taylor's series expansion of $f(z)$ in power of $(z-a)$ (or) taylor's series expansion of $f(z)$ about $z=a$ (around $z=a$)

Maclaurin's series: Taylor's series expansion about $a=0$ is called maclaurin's series ie

$$f(z) = f(0) + (z) f'(0) + \frac{z^2}{2!} f''(0) + \cdots + \frac{z^n}{n!} f^n(0) + \cdots$$

which is called maclaurin's theorem.

Note: suppose we want taylor's series expansion of $f(z)$ around $z=a$. then $f(z)$ must be analytic at $z=a$. & $f(z)$ is analytic within circle $c: |z-a| = R$, where R is as large as possible.

Expansions of some standard functions:

1) $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\forall z$ ie $|z| < \infty$

2) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$

3) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$

4) $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

Pf of 1): Let $f(z) = e^z$ $\Rightarrow f(0) = e^0 = 1$ $\quad f'(z) = e^z$, $f'(0) = e^0 = 1$ $\quad f''(z) = e^z$, $f''(0) = e^0 = 1$

W.k.t maclaurin series $f(z) = f(0) + (z) f'(0) + \frac{z^2}{2!} f''(0) + \cdots \Rightarrow e^z = 1 + z + \frac{z^2}{2!} + \cdots$

Important Note: To obtain taylor's series expansion of $f(z)$ around / about $z=a$, then put $z-a=w$. then

$$f(z)=f(w+a)=\phi(w) \text{ [say]}.$$

Now write the maclaurin's series expansion of $\phi(w)$.

finally substitute $w=z-a$, then we get required taylor's series.

formulae.. $(1+x)^{-1}=1-x+x^2-x^3+x^4-\cdots$ if $|x|<1$ $\quad (1-x)^{-1}=1+x+x^2+\cdots$ if $|x|<1$

problems on taylor's series Expansion of $f(z)$:

prob ①. Expand e^z as taylor's series about $z=1$

soln. Given $f(z)=e^z$, $z=1$

let $z-1=w \Rightarrow z=1+w$

Now $f(z)=f(1+w)=e^{1+w}=e\cdot e^w=\phi(w)$ (say)

$\therefore \phi(w)=e\cdot e^w$

Now write maclaurin's series for $\phi(w)$

ie $\phi(w)=\phi(0)+w\phi'(0)+\frac{w^2}{2!}\phi''(0)+\cdots$

$\phi(w)=e\cdot e^w$, $\quad \phi'(w)=e e^w$, $\quad \phi''(w)=e\cdot e^w$

$\phi(0)=e$, $\quad \phi'(0)=e$ $\quad \phi'(0)=e$

$\therefore \phi(w)=e+ew+\frac{w^2}{2!}e+\cdots$

$\phi(w)=e\left[1+w+\frac{w^2}{2!}+\cdots\right]$

Now replace w by $z-1$

$\Rightarrow \phi(z-1)=e\left[1+(z-1)+\frac{(z-1)^2}{2!}+\cdots\right]$

$\Rightarrow f(z)=e\left[1+(z-1)+\frac{(z-1)^2}{2!}+\cdots\right]$

which is taylors series for $f(z)=e^z$ about $z=1$

(OR)

(2) Expand $f(z) = \frac{z-1}{z+1}$ in taylor's series about the point

(i) $z=0$ (ii) $z=1$.

Sol. (i) Given $f(z) = \frac{z-1}{z+1} = \frac{z+1-2}{z+1} = 1 - \frac{2}{z+1} = 1-2(1+z)^{-1}$

$= 1-2(1-z+z^2-z^3+\cdots)$ if $|z|<1$

$= -1+2(z-z^2+z^3+\cdots)$ if $|z|<1$

$(1+x)^{-1} = 1-x+x^2-x^3+\cdots$ for $|x|<1$. Exp abt '0'

which is a taylors series exp of $f(z)$ about $z=0$.

(ii) $f(z) = \frac{z-1}{z+1}$, $z=1$

Let $z-1=w \Rightarrow z=1+w$

$\Rightarrow f(z) = \frac{w}{w+2} = \frac{w}{2(1+\frac{w}{2})} = \frac{w}{2}\left(1+\frac{w}{2}\right)^{-1}$

$= \frac{w}{2}\left[1-\frac{w}{2}+\left(\frac{w}{2}\right)^2-\left(\frac{w}{2}\right)^3+\cdots\right]$ if $\left|\frac{w}{2}\right|<1$

Exp abt $(w-0)$ ie $z-1-0 = z-1$

$= \frac{w}{2}-\left(\frac{w}{2}\right)^2+\left(\frac{w}{2}\right)^3-\left(\frac{w}{2}\right)^4+\cdots$ if $|w|<2$

$(1+x)^{-1} = 1-x+x^2-x^3+\cdots$ if $|x|<1$

$= \frac{z-1}{2} - \frac{(z-1)^2}{4} + \frac{(z-1)^3}{8} - \frac{(z-1)^4}{16} + \cdots$ if $|z-1|<2$

$f(z) = \sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{z-1}{2}\right)^n$ if $|z-1|<2$

which is a taylors series expansion of $f(z)$ about $z=1$

& it is valid only in the circle of radius '2' & centre (1, 0)

(3) find the taylor's expansion of the function $f(z) = \frac{1}{(1+z)^2}$ with centre at $-i$ (OR) find taylors series exp of $f(z) = \frac{1}{(1+z)^2}$ around $-i$

Sol. Given $f(z) = \frac{1}{(1+z)^2}$ & $a=-i$

$z=a=-i$

(3) Find Taylors series of $f(z) = \frac{1}{(1+z)^2}$ about $z=-i$

w.k.t ~~taylor's theorem~~ for $f(z)$ is

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^n(a) + \cdots$$

~~put~~ $a=-i$

$$\Rightarrow f(z) = f(-i) + (z+i)f'(-i) + \frac{(z+i)^2}{2!}f''(-i) + \cdots + \frac{(z+i)^n}{n!}f^n(-i) + \cdots \quad (1)$$

$$f(z) = \frac{1}{(1+z)^2} \Rightarrow f(-i) = \frac{i}{2}$$

$$f'(z) = \frac{-2}{(1+z)^3} \Rightarrow f'(-i) = \frac{(-1)\ 2!}{(1-i)^3}$$

$$f''(z) = \frac{+6}{(1+z)^4} \Rightarrow f''(-i) = \frac{(3!)}{(1-i)^4}$$

sub $f(z)$, all above values in (1) then

$$f(z) = \frac{i}{2} \mp (z+i)\frac{2!}{(1-i)^3} + \frac{3!}{(1-i)^4}\cdot\frac{(z+i)^2}{2!} + \cdots$$

(4) find the Taylor's

(5) Expand $f(z) = \cos z$ in taylor's series about $z = \pi i$

<u>soln</u>: Given $f(z) = \cos z$, $a = \pi i$

w.k.t taylor's series Expansion of $f(z)$ about $z = a$ is

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \frac{(z-a)^4}{4} f^{iv}(a) + \cdots$$

$$f(z) = f(i\pi) + (z-\pi i) f'(\pi i) + \frac{(z-\pi i)^2}{2!} f''(\pi i) + \frac{(z-\pi i)^3}{3!} f'''(\pi i) + \frac{(z-\pi i)^4}{4!} f^{iv}(\pi i) + \cdots \quad (1)$$

$f(z) = \cos z$

$f(\pi i) = \cos i\pi = \cosh\pi = \frac{e^{\pi} + e^{-\pi}}{2} = -1$

$f'(z) = -\sin z \Rightarrow f'(\pi i) = -\sin \pi i = -i\sinh\pi = 0$

$f''(z) = -\cos z \Rightarrow f''(\pi i) = 1$

$f'''(z) = \sin z \Rightarrow f'''(\pi i) = 0$, $f^{iv}(z) = -\cos z \Rightarrow f^{iv}(\pi i) = -\cos \pi i = -1$

Sub all above values in (1)

$$\Rightarrow \cos z = -1 + (z-\pi i) \times 0 + \frac{(z-\pi i)^2}{2!} \times 1 + \frac{(z-\pi i)^3}{3!} \times 0 + \frac{1}{4!}(z-\pi i)^4(-1) + \cdots$$

$$\Rightarrow \cos z = -1 + \frac{1}{2}(z-\pi i)^2 - \frac{1}{4!}(z-\pi i)^4 + \cdots$$

(6) Expand $\frac{z}{(z+1)(z-2)}$ about $z = 1$ (OR)

write the taylor's series expansion of $\frac{z}{(z+1)(z-2)}$ about $z = 1$

<u>Sol</u>: Given $f(z) = \frac{z}{(z+1)(z-2)}$ & $a = 1$

$$\frac{z}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2} \quad \text{(By partial fractions)}$$

$$\frac{z}{\cancel{(z+1)(z-2)}} = \frac{A(z-2) + B(z+1)}{\cancel{(z+1)(z-2)}} \Rightarrow z = A(z-2) + B(z+1)$$

put $z = 2$ on B.S

$\Rightarrow 2 = 3B \Rightarrow \boxed{B = 2/3}$

now put $z = -1$ on B.S $\Rightarrow -1 = -3A \Rightarrow \boxed{A = 1/3}$

$\therefore \dfrac{z}{(z+1)(z-2)} = \dfrac{2}{3(z+1)} + \dfrac{1}{3(z-2)}$

$\therefore f(z) = \dfrac{2}{3(z+1)} + \dfrac{1}{3(z-2)}$ Now let $z-1=w$ | $\because z=1$

$= \dfrac{2}{3(w+2)} + \dfrac{1}{3(w-1)}$ $\Rightarrow z=1+w$

$= \dfrac{1}{3}\left[1+\dfrac{w}{2}\right]^{-1} + \dfrac{1}{3\left[1-\frac{w}{2}\right]}$

$= \dfrac{1}{3}\left[1+\dfrac{w}{2}\right]^{-1} - \dfrac{1}{3}\left[1-w\right]^{-1}$

$= \dfrac{1}{3}\left[1-\dfrac{w}{2}+\dfrac{w^2}{4}-\dfrac{w^3}{8}+\cdots\right] - \dfrac{1}{3}\left[1+\dfrac{w}{2}+\dfrac{w^2}{4}+\dfrac{w^3}{8}+\cdots\right]$ if $\left|\frac{w}{2}\right|<1$ & $|w|<1$ $\Rightarrow |w|<2$ & $|w|<1$ $\Rightarrow |w|<1$ (Common Region)

$\left(\because (1-z)^{-1} = 1+z+z^2+z^3+\cdots\right.$
$\left.(1+z)^{-1} = 1-z+z^2-z^3+\cdots\right)$

$f(z) = \dfrac{1}{3}\left[1-\dfrac{1}{2}(z-1)+\dfrac{1}{4}(z-1)^2-\dfrac{1}{8}(z-1)^3+\cdots\right] - \dfrac{2}{3}\left[1+(z-1)+(z-1)^2+(z-1)^3+\cdots\right]$

ie this series is valid in the region $|z-1|<1$ if $|z-1|<1$

(7) Expand ze^z by taylor's series about $z=1$

Soln Given $f(z) = ze^z$, & $a=1$

put $z-1=w \Rightarrow z=1+w$

$\therefore f(z) = (1+w)e^{1+w} = e(1+w)e^w$

$= e(1+w)\left[1+w+\dfrac{w^2}{2!}+\dfrac{w^3}{3!}+\cdots\right]$

$= ez\left[1+(z-1)+\dfrac{(z-1)^2}{2!}+\dfrac{(z-1)^3}{3!}+\cdots\right]$

(8) find the taylor's series for $\dfrac{z}{z+2}$ about $z=1$. also find the region of convergence.

Soln Given $f(z) = \dfrac{z}{z+2}$

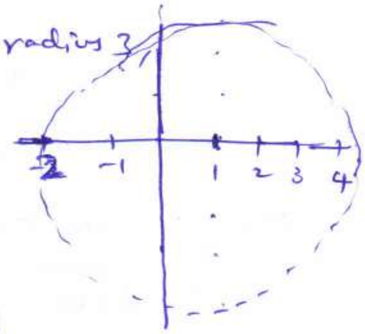
the singular point is $z+2=0 \Rightarrow z=-2$

If the centre of the circle is at $z=1$, then

distance from singularity $z=-2$ to centre $z=1$ is 3 units.

hence, if a circle is drawn with centre $z=1$ & radius 3

then the within the circle $|z-1|=3$, the given function $f(z)$ is analytic.



Therefore, the given function can be expanded in Taylor's series within the circle $|z-1|=3$.

$\therefore$ $|z-1|=3$ is the circle of convergence.

Now $f(z) = \dfrac{z}{z+2} = A + \dfrac{B}{z+2}$ (By partial fraction)

$$\Rightarrow \frac{z}{z+2} = \frac{A(z+2)+B}{(z+2)}$$

$$\Rightarrow z = A(z+2)+B$$

put $z=-2$ then $\boxed{B=-2}$

comparing 'z' coefficient on B.S

$\Rightarrow \boxed{A=1}$

$\therefore$

$$f(z) = 1 - \frac{2}{z+2}$$

$$= 1 - \frac{2}{(z-1)+3}$$

$$= 1 - \frac{2}{3}\left[\frac{1}{1+\frac{(z-1)}{3}}\right]$$

$$= 1 - \frac{2}{3}\left[1+\left(\frac{z-1}{3}\right)\right]^{-1}$$

$$f(z) = 1 - \frac{2}{3}\left[1 - \frac{z-1}{3} + \left(\frac{z-1}{3}\right)^2 - \left(\frac{z-1}{3}\right)^3 + \cdots\right]$$ is valid if $\left|\frac{z-1}{3}\right|<1$ $\Rightarrow |z-1|<3$

($\because (1+x)^{-1} = 1-x+x^2+x^3+\cdots$ is valid if $|x|<1$)

(9) Expand log z by taylor's series about $z=1$

Sol^n Given $f(z) = \log z$, $z=1$

let $z-1=w \Rightarrow z=1+w$

$\Rightarrow f(1+w) = \log(1+w) = f(w)$

Now write the maclaurin's series for $f(w)$

ie $f(w) = f(0) + w f'(0) + \frac{w^2}{2!} f''(0) + \cdots$ —(1) if $|w|<1$

$f(w) = \log(1+w)$, $f'(w) = \frac{1}{1+w}$ $f''(w) = \frac{-1}{(1+w)^2}$

$f(0) = \log 1 = 0$ $f'(0) = 1$, $f''(0) = -1$

$\therefore f(w) = w + \frac{w^2}{2!}(-1) + \cdots$

$\log(1+w) = (z-1) - \frac{(z-1)^2}{2!} + \frac{1}{3}(z-1)^3 - \cdots$

$\Rightarrow \log z = (z-1) - \frac{(z-1)^2}{2!} + \frac{1}{2}(z-1)^3 - \cdots$ for $|z-1|<1$

(10) obtain the Expansion of $\frac{1}{(z-1)(z-3)}$ in a taylor's series in powers of $(z-4)$ and determine the region of convergence.

Sol^n. Given let $f(z) = \frac{1}{(z-1)(z-3)}$ & $z=4$.

The singular points of $f(z)$ are: $(z-1)(z-3)=0$

$\Rightarrow z = 1, 3$

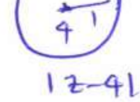
If the centre of the circle is $z=4$, then

the distances from centre $z=4$ to the singular points $1, 3$ are 3 & 1.

$\therefore$ we have to consider a circle with centre '4' & radius 1

then $f(z)$ is analytic within the circle $|z-4|=1$

hence $f(z)$ can be Expanded in taylor's series within $|z-4|=1$



$\therefore$ $|z-4|=1$ is called circle of convergence.

$|z-4|=1$

6 6

$$f(z) = \frac{1}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3} \quad (\because \text{ by partial fractions})$$

$$\Rightarrow \frac{1}{(z-1)(z-3)} = \frac{A(z-3)+B(z-1)}{(z-1)(z-3)}$$

$$\Rightarrow 1 = A(z-3) + B(z-1)$$

put $z=3$ on B.S $\Rightarrow 2B = 1 \Rightarrow B = \frac{1}{2}$

Now put $z=1$ on B.S $\Rightarrow -2A = 1 \Rightarrow A = \frac{-1}{2}$

$$\Rightarrow f(z) = \frac{-1}{2(z-1)} + \frac{1}{2(z-3)}$$

Let $z-4 = w \Rightarrow z = w+4$ $\quad (\because z=4$ is given$)$

Intersection $|w|<1$

$$\Rightarrow f(z) = \frac{-1}{2(w+3)} + \frac{1}{2(w+1)}$$

$|w|<1$
$|w|<3$
$\Rightarrow |w|<1$

$$\Rightarrow f(z) = \frac{-1}{6}\left[1+\frac{w}{3}\right]^{-1} + \frac{1}{2}\left[1+w\right]^{-1}$$

$$\Rightarrow f(z) = -\frac{1}{6}\left[1-\frac{w}{3}+\left(\frac{w}{3}\right)^2-\left(\frac{w}{3}\right)^3+\cdots\right] + \frac{1}{2}\left[1-w+w^2-w^3+\cdots\right] \quad \text{if } |w|<1 \;\&\; \left|\frac{w}{3}\right|<1$$

$$\Rightarrow f(z) = -\frac{1}{6}\left[1-\frac{(z-4)}{3}+\left(\frac{z-4}{3}\right)^2-\left(\frac{z-4}{3}\right)^3+\cdots\right] + \frac{1}{2}\left[1-(z-4)+(z-4)^2-(z-4)^3+\cdots\right]$$

if $|z-4|<1$

$$= \left(\frac{1}{2}-\frac{1}{6}\right) + \left(-\frac{1}{2}+\frac{1}{18}\right)(z-4) + \left(\frac{1}{2}-\frac{1}{54}\right)(z-4)^2 + \cdots$$

$$f(z) = \frac{1}{3} - \frac{4}{9}(z-4) + \frac{13}{27}(z-4)^2 - \cdots$$

[OR] for $f(z)$ Taylor series (OR) Maclaurin series $f(w)$

(11) Expand $f(z) = \dfrac{1}{z^2-z-6}$ about (i) $z=-1$ (ii) $z=1$

Sol. (i) Given $f(z) = \dfrac{1}{z^2-z-6}$ & $z=-1$

$$= \frac{1}{(z-3)(z+2)}$$

$$= \frac{1}{5}\left[\frac{1}{z-3} - \frac{1}{z+2}\right] \quad (\because \text{ By partial fractions})$$

Let $z+1=w \Rightarrow z = w-1$

common Region

$|w|<1$ is common region

$\Rightarrow f(z) = \frac{1}{5}\left[\frac{1}{w-4} - \frac{1}{w+1}\right] = \frac{1}{5}\cdot\frac{1}{(w-4)} - \frac{1}{5}\cdot\frac{1}{(w+1)}$

$= \cancel{\frac{1}{5}\Big[} = \frac{1}{-20}\left[1-\frac{w}{4}\right]^{-1} - \frac{1}{5}\left[1+w\right]^{-1}$

$= -\frac{1}{20}\left[1+\left(\frac{w}{4}\right)+\left(\frac{w}{4}\right)^2+\left(\frac{w}{4}\right)^3+\cdots\right] - \frac{1}{5}\left[1-w+w^2-w^3+\cdots\right]$ if $|w|<1$ & $\left|\frac{w}{4}\right|<1$

$\Rightarrow |w|<1$ & $|w|<4$
$\Rightarrow |w|<1$

$f(z) = -\frac{1}{20}\left[1+\frac{z+1}{4}+\left(\frac{z+1}{4}\right)^2+\left(\frac{z+1}{4}\right)^3+\cdots\right] - \frac{1}{5}\left[1-(z+1)+(z+1)^2-(z+1)^3+\cdots\right]$ if $|z+1|<1$

$f(z) = -\frac{1}{20}\sum_{n=0}^{\infty}\left(\frac{z+1}{4}\right)^n - \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n(z+1)^n$ which is required taylors series.

& it is valid in $|z+1|<1$

H/w : (1) Expand $\log(1-z)$ when $|z|<1$ using taylor's series.

(2) find taylor's series expansion of $f(z) = \frac{2z^3+1}{z^2+z}$ about the point
(i), $z=i$, (ii), $z=1$

(3) Expand $f(z) = \sin z$ in taylor's series about (i), $z=\frac{\pi}{4}$ (ii), $z=\frac{\pi}{2}$

(4) Obtain the taylor's series expansion of $f(z) = \frac{e^z}{z(z+1)}$ about $z=2$

soln of (1). let $z-0=w \Rightarrow z=w$ $\therefore \varphi(w) = \log(1-w)$

soln (2) $\frac{2z^3+1}{z(z+1)}$ $f(z) = f(i) + (z-i)f'(i) + \cdots$

soln of (3) : $f(z) = f(\pi/4) + \left(z-\frac{\pi}{4}\right)f'(\pi/4) + \frac{(z-\frac{\pi}{4})^2}{2!}f''(\pi/4) + \cdots$

soln of (4) : $f(z) = f(2) + (z-2)f'(2) + \frac{(z-2)^2}{2!}f''(2) + \cdots$

<u>Laurent's</u> <u>series</u> <u>Expansion</u>: we have seen under taylor's series that if $f(z)$ is analytic at $z=a$, we can have a series expansion of $f(z)$ in non-negative powers of $(z-a)$ which is valid in a region given by $|z-a|<R$. for suitable R. A natural question that arise is: "can we think of having a power series expansion around $z=a$ even if it is a singular point of $f(z)$?" Some of the examples worked out earlier do indicate that such expansion are possible. However in this case, the region of validity of the expansion is to be carefully written.

Laurent's theorem gives a procedure to expand a given function in powers of $(z-a)$. The series expansion may have positive as well as negative powers.

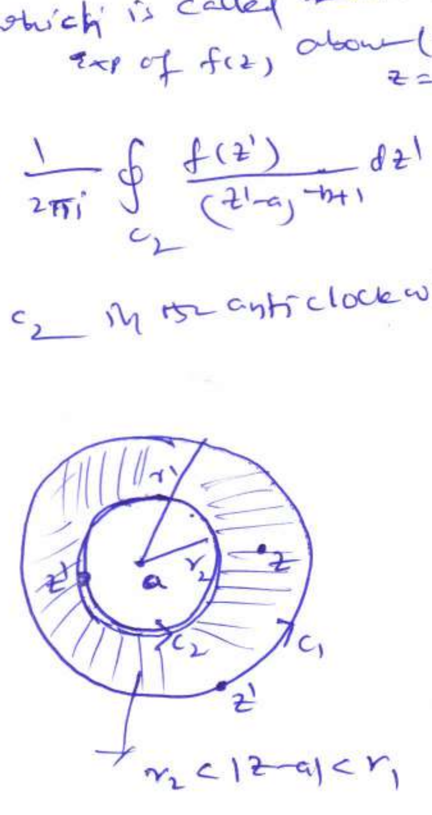
<u>Laurent's theorem</u>: Let c_1 & c_2 be two circles given by $|z'-a|=r_1$ & $|z'-a|=r_2$ respectively where $r_2<r_1$, & z'

Let $f(z)$ be analytic on c_1 & c_2 & throughout the region between the two circles. Let 'z' be any point in the ring shaped region between the two circles c_1 & c_2. then

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$ which is called Laurent series exp of $f(z)$ about (around) $z=a$.

where $a_n=\dfrac{1}{2\pi i}\displaystyle\oint_{c_1}\frac{f(z')}{(z'-a)^{n+1}}dz'$ & $b_n=\dfrac{1}{2\pi i}\displaystyle\oint_{c_2}\frac{f(z')}{(z'-a)^{-n+1}}dz'$

where the integrals are taken around c_1 & c_2 in the anticlockwise sense.



Prob ①. Let $f(z) = \dfrac{1}{(1-z)(z-2)}$. Then find

i) Laurent series exp in the annular region $1<|z|<2$

ii) Laurent series exp in $|z|>2$

$f(z) = \dfrac{1}{z-1} - \dfrac{1}{z-2}$ (By partial fraction)

$1<|z|$ $\quad |z|<2$

$\dfrac{1}{|z|}<1$ $\quad \dfrac{|z|}{2}<1$

(i) $\dfrac{1}{z(1-\frac{1}{z})}$ $\quad \dfrac{1}{z(1-\frac{2}{z})}$

(2) Find Laurent's series for $f(z) = \dfrac{1}{z^2(1-z)}$ & find the region of convergence. [OR] find two Laurent's series expansions in powers of z for $f(z) = \dfrac{1}{z^2(1-z)}$ & specify the regions in which these expansions are valid.

Solution:

Given $f(z) = \dfrac{1}{z^2(1-z)}$

The singular points are $z=0$ & $z=1$

Now $f(z) = \dfrac{1}{z^2(1-z)} = \dfrac{1}{z^2}(1-z)^{-1}$

$= \dfrac{1}{z^2}\left[1+z+z^2+\cdots\right]$ valid only if $z \neq 0$, & $|z|<1$

$f(z) = \dfrac{1}{z^2} + \dfrac{1}{z} + 1 + z + z^2 + \cdots$ is valid only if $0<|z|<1$

$f(z) = \sum\limits_{n=0}^{\infty} z^{n-2}$ if $0<|z|<1$ $\Rightarrow f(z) = \sum\limits_{n=0}^{\infty}(z-0)^{n-2}$ if $0<|z|<1$

only analytic part is there & principal part is not there

which is one Laurent's series expansion in powers of z. (ie $(z-0)$)

$f(z) = \dfrac{1}{z^2(1-z)} = \dfrac{-1}{z^2(z-1)}$

$= \dfrac{-1}{z^2 \cdot z\left(1-\frac{1}{z}\right)} = \dfrac{-1}{z^3\left(1-\frac{1}{z}\right)} = \dfrac{-1}{z^3}\left(1-\dfrac{1}{z}\right)^{-1}$

$= \dfrac{-1}{z^3}\left[1+\dfrac{1}{z}+\dfrac{1}{z^2}+\cdots\right]$ valid only if $z \neq 0$ & $\left|\dfrac{1}{z}\right|<1$

$= -\left(\dfrac{1}{z^3}+\dfrac{1}{z^4}+\dfrac{1}{z^5}+\cdots\right)$ if $|z|>1$

$f(z) = -\sum\limits_{n=0}^{\infty} z^{-n-3}$ if $|z|>1$ $\Rightarrow f(z) = -\sum\limits_{n=0}^{\infty}(z-0)^{-n-3}$ if $|z|>1$

only principal part & analytic part is not there

This is another Laurent's series expansion in power of z

(3) Expand $f(z) = \dfrac{1}{z^2-3z+2}$ in the region (i) $1<|z|<2$ ~~$0<|z-1|<1$~~

(ii) ~~$1<|z|<2$~~ $0<|z-1|<1$

Soln

$f(z) = \dfrac{1}{z^2-3z+2} = \dfrac{1}{(z-1)(z-2)} = \dfrac{A}{z-1} + \dfrac{B}{z-2}$ ($\because$ By partial fractions)

$\Rightarrow \dfrac{1}{\cancel{(z-1)(z-2)}} = \dfrac{A(z-2)+B(z-1)}{\cancel{(z-1)(z-2)}}$

$\Rightarrow 1 = A(z-2) + B(z-1)$

put $z=1$ on B.S then $\boxed{A=-1}$

put $z=2$ on B.S then $\boxed{B=1}$

$$\therefore f(z) = \frac{-1}{z-1} + \frac{B}{z-2}$$

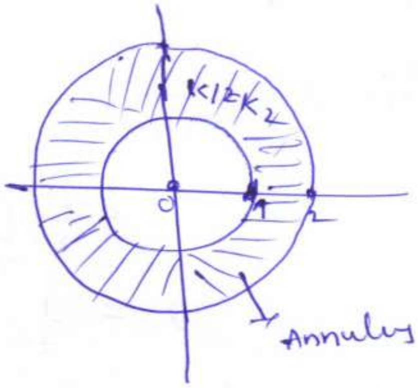
$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

The singular points of $f(z)$ are $z=1,2$

(i) consider $1<|z|<2$

ie $1<|z|$, $|z|<2$

$\Rightarrow \left|\frac{1}{z}\right|<1 \quad \left|\frac{z}{2}\right|<1$



$f(z)$ is analytic in the Annulus ie in the region $1<|z|<2$

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{2}\left[1-\frac{z}{2}\right]^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right]$$

valid only if $\left|\frac{1}{z}\right|<1$ & $\left|\frac{z}{2}\right|<1$

$\Rightarrow 1<|z|<2$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \quad \text{if } 1<|z|<2$$

$$f(z) = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} \quad \text{if } 1<|z|<2$$

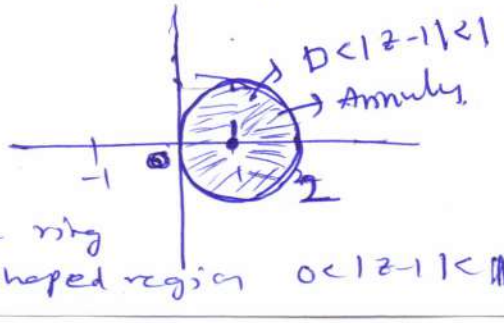
$$f(z) = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z-0}{2}\right)^n - \sum_{n=0}^{\infty}(z-0)^{-n-1}$$ Analytic part + principal part

which is the Laurent series expansion of $f(z)$ about 0 (or in powers of z) in the region $1<|z|<2$

(ii) consider $0<|z-1|<1$

we have $f(z) = \frac{-1}{(z-1)} + \frac{1}{z-2}$

The function $f(z)$ is analytic in the ring shaped region $0<|z-1|<1$



$$= \frac{-1}{(z-1)} + \frac{1}{(z-1)-1}$$

$$= -\frac{1}{(z-1)} + \frac{1}{-[1-(z-1)]}$$

$$= \frac{-1}{(z-1)} - [1-(z-1)]^{-1}$$

$$= \frac{-1}{(z-1)} - [1+(z-1)+(z-1)^2+\cdots] \quad \text{valid only if } |z-1|<1$$

$$= -(z-1)^{-1} - \sum_{n=0}^{\infty}(z-1)^n$$

$$f(z) = \underbrace{-\sum_{n=0}^{\infty}(z-1)^n}_{\text{Analytic part}} + \underbrace{-(z-1)^{-1}}_{\text{principal part}}$$

Note: Here the principal part contains finite no. of terms (only one term)

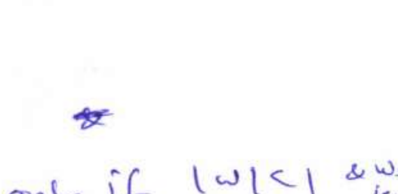
which is a Laurent series expansion of $f(z)$ about $z=1$ in the region $0<|z-1|<1$

(OR)

(ii) consider $0<|z-1|<1$, we have $f(z) = \frac{-1}{(z-1)} + \frac{1}{z-2}$

singular points of $f(z)$ are $z=1,2$

The function $f(z)$ is analytic in the ring shaped (Annulus) region $0<|z-1|<1$



Put $z-1=w \Rightarrow z=1+w$.

$$\therefore f(z) = \frac{-1}{w} + \frac{1}{w-1}$$

$$= \frac{-1}{w} - \frac{1}{(1-w)}$$

$$= \frac{-1}{w} - (1-w)^{-1}$$

$$= \frac{-1}{w} - (1+w+w^2+w^3+\cdots) \quad \text{valid only if } |w|<1 \;\&\; w\neq 0 \text{ ie } w\neq 0 \text{ ie } 0<|w|<1$$

$$= \frac{-1}{z-1} - [1+(z-1)+(z-1)^2+(z-1)^3+\cdots] \quad \text{if } 0<|z-1|<1$$

$$= \frac{-1}{z-1} - \sum_{n=0}^{\infty}(z-1)^n$$

$$f(z) = \underbrace{-\sum_{n=0}^{\infty}(z-1)^n}_{\text{Analytic part}} + \underbrace{-(z-1)^{-1}}_{\text{Principal part}}$$

which is a Laurent exp of $f(z)$ about $z=1$ in the region $0<|z-1|<1$ [ie in powers of $z-1$]

(4) Find the Laurent Series expansion of the function

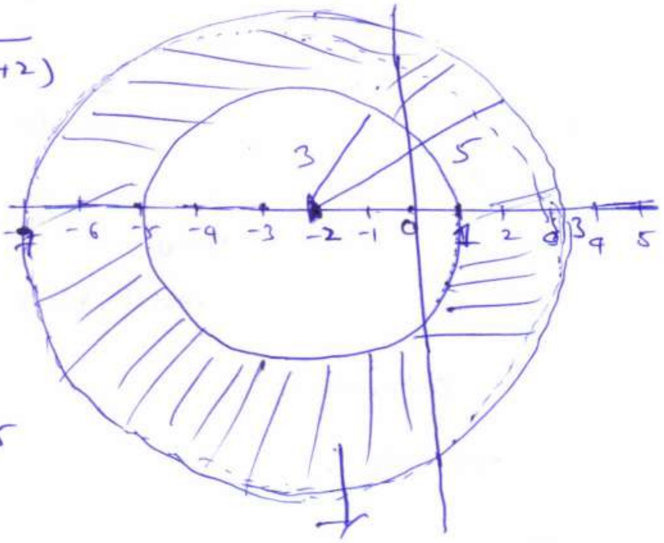
$f(z) = \dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3<|z+2|<5$

(OR) Expand $f(z)$ in Annular region $3<|z+2|<5$

Soln: $f(z)=\dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)} = \dfrac{A}{(z-1)}+\dfrac{B}{(z-3)}+\dfrac{C}{z+2}$ ($\because$ By partial fractions)

$A=1,\ B=-1,\ C=1$

$\therefore f(z) = \dfrac{1}{(z-1)} - \dfrac{1}{(z-3)} + \dfrac{1}{(z+2)}$



The singular points of $f(z)$ are

$z=1,-2,3$

Clearly the function $f(z)$ is analytic in the region $3<|z+2|<5$

It is required to find Laurent series expansion $3<|z+2|<5$ of $f(z)$ about (around) $z=-2$ in the region $3<|z+2|<5$

Let $z+2=w \Rightarrow z=w-2$

$\therefore f(z) = \dfrac{1}{w-3} - \dfrac{1}{w-5} + \dfrac{1}{w}$ & the region is $3<|w|<5$

$3<|w|$ & $|w|<5$

$\Rightarrow \frac{3}{|w|}<1,\ \frac{|w|}{5}<1$

$= \frac{1}{w}\left[1-\frac{3}{w}\right]^{-1} + \frac{1}{5}\left[1-\frac{w}{5}\right]^{-1} + w^{-1}$

$= \frac{1}{w}\left[1+\frac{3}{w}+\left(\frac{3}{w}\right)^2+\left(\frac{3}{w}\right)^3\right] + \frac{1}{5}\left[1+\frac{w}{5}+\left(\frac{w}{5}\right)^2+\left(\frac{w}{5}\right)^3+\cdots\right] + w^{-1}$

valid only if $\left|\frac{3}{w}\right|<1$ & $\left|\frac{w}{5}\right|<1$ & $w\neq 0$

$\Rightarrow 3<|w|$ & $|w|<5$

$\Rightarrow 3<|w|<5$

$= \cancel{\frac{1}{3}}\Big[$

$f(z) = \frac{1}{w}\left[1+\frac{3}{z+2}+\left(\frac{3}{z+2}\right)^2+\left(\frac{3}{z+2}\right)^3+\cdots\right] + \frac{1}{5}\left[1+\left(\frac{z+2}{5}\right)+\left(\frac{z+2}{5}\right)^2+\left(\frac{z+2}{5}\right)^3+\cdots\right]+(z+2)^{-1}$

valid only if $3<|z+2|<5$

$f(z) = \frac{1}{w}\left[\cancel{1}+\cancel{\left(\frac{3}{z+2}\right)}+\cdots+\left(\frac{3}{z+2}\right)^2+\left(\frac{3}{z+2}\right)^3+\cdots\right]+\left[\frac{1}{5}+\frac{1}{5}\left(\frac{z+2}{5}\right)+\frac{1}{5}\left(\frac{z+2}{5}\right)^2+\frac{1}{5}\left(\frac{z+2}{5}\right)^3+\cdots\right]+\frac{1}{z+2}$

$f(z) = \frac{1}{z+2}\sum_{n=0}^{\infty}\left(\frac{3}{z+2}\right)^n + \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{z+2}{5}\right)^n + \frac{1}{z+2} = \frac{1}{5}\sum_{n=0}^{\infty}\frac{1}{5^n}(z+2)^n + \frac{1}{(z+2)}\sum_{n=0}^{\infty}3^n(z+2)^{-n} + \frac{1}{z+2}$

AP + PP

[~~OR~~]

$$f(z) = \frac{1}{z+2}\sum_{n=0}^{\infty}\left(\frac{3}{z+2}\right)^n + \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{z+2}{5}\right)^n + \frac{1}{z+2}$$ valid only if $3<|z+2|<5$

(OR)

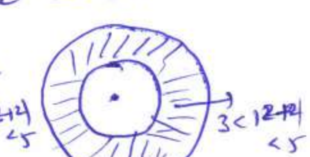
Given

$$f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$$ & the region is $3<|z+2|<5$

$$= \frac{A}{(z-1)} + \frac{B}{(z-2)} + \frac{C}{z+2}$$ (∵ By partial fractions)

The singular points of $f(z)$ are $z=1,3,-2$

$f(z)$ is analytic in $3<|z+2|<5$



$A=1, \quad B=-1, \quad C=1$

$$\therefore f(z) = \frac{1}{(z-1)} - \frac{1}{(z-3)} + \frac{1}{z+2}$$

$$= \frac{1}{(z+2)-3} - \frac{1}{(z+2)-5} + \frac{1}{z+2}$$

$$= \frac{1}{(z+2)\left[1-\frac{3}{(z+2)}\right]} - \frac{1}{-5\left[1-\frac{z+2}{5}\right]} + \frac{1}{z+2}$$

$3<|z+2|<5$

$\cancel{\frac{3}{|z+2|}}$

$3<|z+2|, \quad |z+2|<5$

$\frac{3}{|z+2|}<1 \quad \frac{|z+2|}{5}<1$

$$= \frac{1}{z+2}\left[1-\frac{3}{z+2}\right]^{-1} + \frac{1}{5}\left[1-\frac{z+2}{5}\right]^{-1} + \frac{1}{z+2}$$

$$= \frac{1}{z+2}\left[1+\frac{3}{z+2}+\left(\frac{3}{z+2}\right)^2+\cdots\right] + \frac{1}{5}\left[1+\frac{z+2}{5}+\left(\frac{z+2}{5}\right)^2+\cdots\right] + \frac{1}{z+2}$$

valid only if $\left|\frac{3}{z+2}\right|<1$

$\Rightarrow 3<|z+2|$

& $\left|\frac{z+2}{5}\right|<1$

$\Rightarrow |z+2|<5$

ie $3<|z+2|<5$

$$= \left[\frac{2}{z+2} + \frac{3}{(z+2)^2} + \frac{9}{(z+2)^3} + \cdots\right] + \frac{1}{5}\left[1+\frac{z+2}{5}+\left(\frac{z+2}{5}\right)^2+\cdots\right]$$

$$f(z) = \left[2(z+2)^{-1} + 3(z+2)^{-2} + 9(z+2)^{-3} + \cdots\right] + \frac{1}{5}\left[1+\frac{1}{5}(z+2)+\frac{1}{5^2}(z+2)^2+\cdots\right]$$

principal part + Analytical part

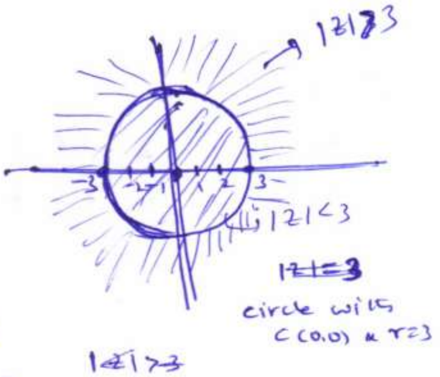
which is a Laurent series exp of $f(z)$ in power of $(z+2)$ or around $z=-2$

(5) Expand the Laurent series of $\frac{z^2-1}{(z+2)(z+3)}$ for $|z|>3$

Soln. Given $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ & the region is $|z|>3$

The singular points of $f(z)$ are $z=-2,-3$

The function $f(z)$ is analytic in the region $|z|>3$



Now $f(z)=\frac{z^2-1}{(z+2)(z+3)} = C + \frac{A}{(z+2)} + \frac{B}{z+3}$ ($\because$ By partial fractions)

$z^2-1 = A(z+3) + B(z+2) + C(z+2)(z+3)$

put $z=-3$ on B.S then $-B = 8 \Rightarrow \boxed{B=-8}$

put $z=-2$ on B.S then $A = 3 \Rightarrow \boxed{A=3}$

comparing z coeff on B.S then $A+B+5C=0 \Rightarrow \boxed{C=1}$

$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

It is required to find Laurent series expansion of $f(z)$ about $z=0$ in the region $|z|>3$

$|z|>3>2$

$\Rightarrow 2<3<|z|$

$2<|z|$ & $\frac{3}{|z|}<1$ $3<|z|$

$\Rightarrow \frac{2}{|z|}<1$ & $\frac{3}{|z|}<1$

common region of $2<|z|$ & $3<|z|$ is $|z|>3$

$\therefore f(z) = 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{z(1+\frac{3}{z})}$

$1 + \frac{3}{z}\left[1+\frac{2}{z}\right]^{-1} - \frac{8}{z}\left[1+\frac{3}{z}\right]^{-1}$

$f(z) = 1 + \frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right] - \frac{8}{z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\cdots\right]$

valid only if $\left|\frac{2}{z}\right|<1$ & $\left|\frac{3}{z}\right|<1$

$\Rightarrow 2<|z|$ & $3<|z|$

$f(z) = 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^3$ $\Rightarrow |z|>3>2$

only if $|z|>3$

which is Laurent series expan. of $f(z)$ in power of z (ie $(z-0)$)

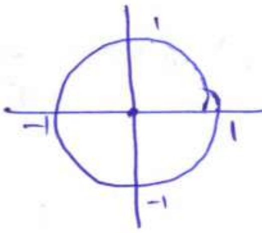
Note: There is no Analytic part in above Laurent series (ie no positive powers of $(z-0)$)

⑥ Expand $\dfrac{1}{z^2(z-3)^2}$ as a Laurent's series a) $|z|<1$ b) $|z|>3$ (11)

Soln. a) Given $f(z) = \dfrac{1}{z^2(z-3)^2}$ & the region is $|z|<1$

The singular points of $f(z)$ are : $z^2(z-3)^2=0$

$\Rightarrow z=0 \quad z=3$



The function $f(z)$ is analytic every where in the region $|z|<1$ Except at $z=0$.

Now $f(z) = \dfrac{1}{z^2(z-3)^2}$

It is required to find Laurent series Expansion of $f(z)$ about $z=0$ in the region $|z|<1$

$$f(z) = \frac{1}{z^2 \cdot 9 \cdot \left[1-\frac{z}{3}\right]^2}$$

$$= \frac{1}{9z^2}\left[1-\frac{z}{3}\right]^{-2}$$

$$= \frac{1}{9z^2}\left[1+\frac{2z}{3}+3\left(\frac{z}{3}\right)^2+4\left(\frac{z}{3}\right)^3+\cdots\right]$$ valid only if $\left|\frac{z}{3}\right|<1$ i.e $|z|<3$ & $z\neq0$

$$f(z) = \frac{1}{9z^2}\sum_{n=0}^{\infty}(n+1)\left(\frac{z}{3}\right)^n$$ valid only if $0<|z|<3$

$\left(\because (1-z)^{-2} = 1+2\cdot z+3\cdot z^2+4\cdot z^3+\cdots\right)$ if $|z|<1$

(Analytic part)

The above series is valid in the region $|z|<3$

$\therefore$ It is also valid in the region $|z|<1$ $\quad (\because |z|<1 \subset |z|<3)$

The above series is called Laurent series around $z=0$ (or) in powers of $(z-0)$ & it consists of only analytic part & there is no principal part (i.e principal part contains zero terms)

b) $f(z) = \dfrac{1}{z^2(z-3)^2}$ & the region is $|z|>3$

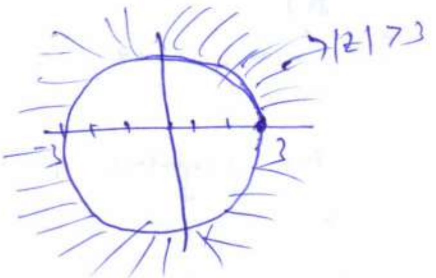
~~The singu~~ ~~It is required~~

The singular points are $z=0$ & 3

$f(z)$ is analytic every where except at $z=0,3$ but 0 & 3 are not included in the region $|z|>3$

$\therefore$ $f(z)$ is analytic everywhere on the region $|z|>3$

$$|z|>3 \Rightarrow \frac{3}{|z|}<1$$



$$\therefore f(z)=\frac{1}{z^2(z-3)^2} = \frac{1}{z^2\cdot z^2\left(1-\frac{3}{z}\right)^2}$$

$$=\frac{1}{z^4}\left[1-\frac{3}{z}\right]^{-2}$$

$$=\frac{1}{z^4}\left[1+2\frac{3}{z}+3\cdot\left(\frac{3}{z}\right)^2+4\left(\frac{3}{z}\right)^3+\cdots\right]$$

$$f(z)=\frac{1}{z^4}\sum_{n=0}^{\infty}(n+1)\left(\frac{3}{z}\right)^n \quad \text{valid only if } \left|\frac{3}{z}\right|<1$$

ie $3<|z| \Rightarrow |z|>3$

(7) Expand $\dfrac{1}{z(z^2-3z+2)}$ in the region (OR) Express $f(z)=\frac{1}{z(z^2-3z+2)}$ in series of +ve & -ve Powers of z (ie $z_0=0$)

a) $1\le|z|\le 2$ b) $0\le|z|\le 1$ c) $|z|\geq 2$

Sol Given $f(z)=\dfrac{1}{z(z^2-3z+2)} = \dfrac{1}{z(z-1)(z-2)} = \dfrac{A}{z}+\dfrac{B}{(z-1)}+\dfrac{C}{(z-2)}$

($\because$ By partial fraction)

$A=\frac{1}{2}$, $B=-1$, $C=\frac{1}{2}$

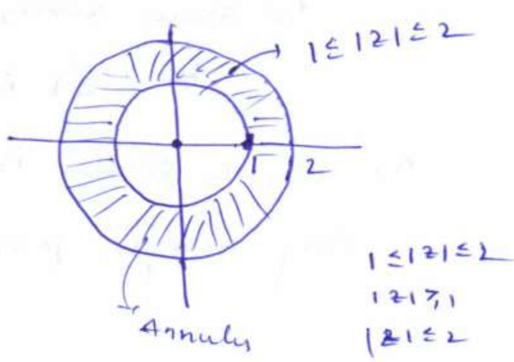
$$\therefore f(z)=\frac{1}{2z}-\frac{1}{z-1}+\frac{1}{2(z-2)}$$

The singular points of $f(z)$ are $z=0,1,2$ $\therefore$ $f(z)$ is analytic everywhere except at $z=0,1,2$

a) $1\le|z|\le 2$

$f(z)$ is analytic in the region $1\le|z|\le 2$

It is required to find Laurent series expansion of $f(z)$ in $1\le|z|\le 2$ Power of $(z-0)$



$\Rightarrow$ $1\le|z|$ & $|z|\le 2$

$\Rightarrow$ $\frac{1}{|z|}\le 1$ & $\frac{|z|}{2}\le 1$

$$f(z)=\frac{1}{2z}-\frac{1}{z-1}+\frac{1}{2(z-2)}$$

$$=\frac{1}{2z}-\frac{1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{4\left(1-\frac{z}{2}\right)}$$

$$=\frac{1}{2z}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}-\frac{1}{4}\left(1-\frac{z}{2}\right)^{-1}$$

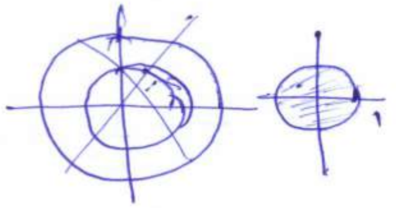
$$=\frac{1}{2z}-\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right]-\frac{1}{4}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

Hence $f(z) = \frac{1}{2}(z-0)^{-1} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n} - \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{n}$.

$f(z) = \frac{1}{2}(z-0)^{-1} - \sum_{n=0}^{\infty} z^{-1}z^{-n} - \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{2^n}(z-0)^n$

$f(z) = \frac{1}{2}(z-0)^{-1} \underbrace{- \sum_{n=0}^{\infty}(z-0)^{-n-1}}_{\text{principal part.}} \underbrace{-\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{2^n}(z-0)^n}_{\text{Analytic part}}$

(ii) $0 \le |z| \le 1$,



$f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$

$= \frac{1}{2z} + \frac{1}{(1-z)} - \frac{1}{4(1-\frac{z}{2})}$

$= \frac{1}{2z} + (1-z)^{-1} - \frac{1}{4}\left(1-\frac{z}{2}\right)^{-1}$

$= \frac{1}{2z} + (1+z+z^2+z^3+\cdots) - \frac{1}{4}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$

valid only if $z \neq 0$ & $|z| \le 1$ & $|\frac{z}{2}| \le 1$ $\Rightarrow 0 \le |z| \le 1$

$f(z) = \frac{1}{2z} - \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n$

& it is a Laurent series exp of $f(z)$ in the annulus $0 \le |z| \le 1$

(iii) $|z| \ge 2$
$\Rightarrow |z| \ge 2 \ge 1 \ge 0$

$|z| \ge 2 \Rightarrow 1 \ge \frac{2}{|z|} \Rightarrow \frac{2}{|z|} \le 1$
$\Rightarrow 1 \le 2 \le |z| \Rightarrow \frac{1}{|z|} \le 1 \quad \frac{2}{|z|} < 1$

$|z| \ge 1$
$\frac{1}{|z|} \le 1$

$\frac{z}{2} \ge 1$
$\frac{2}{z}$
$1 \ge \frac{2}{z}$
$\frac{2}{z} \le 1$
$|\frac{2}{z}| \le 1$

$f(z) = \frac{1}{2z} - \frac{1}{(z-1)} + \frac{1}{2(z-2)}$

$= \frac{1}{2z} - \frac{1}{z(1-\frac{1}{z})} + \frac{1}{2z(1-\frac{2}{z})}$

$= \frac{1}{2z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2z}\left(1-\frac{2}{z}\right)^{-1}$

$= \frac{1}{2z} - \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right] + \frac{1}{2z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right]$

$= \frac{1}{2z} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$

which is a Laurent series in $|z| \ge 2 \Rightarrow 2 \le |z| \le R$.

(8) S.T $(1+z)^{-1} = \sum_{n=0}^{\infty} (-1)^n z^{-n-1}$ for $|z|>1$

Sol

$f(z) = (1+z)^{-1}$ & $|z|>1 \Rightarrow 1<|z| \Rightarrow \frac{1}{|z|}<1$

$= \frac{1}{1+z}$

$= \frac{1}{z(1+\frac{1}{z})}$

$= \frac{1}{z}(1+\frac{1}{z})^{-1} = \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$ valid if $|\frac{1}{z}|<1 \Rightarrow |z|>1$

$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots$ if $|z|>1$

$= \sum_{n=0}^{\infty} (-1)^n z^{-n-1}$ if $|z|>1$

(9) for the function $f(z) = \frac{2z^3+1}{z^2+z}$ Find

a) A Taylor's expansion valid in the nbd of the point i.

b) A Laurent's series valid within the annulus of which centre is origin.

(10) Expand $\dfrac{1}{(z^2+1)(z^2+2)}$ in positive & negative powers of z if $1<|z|<\sqrt{2}$. (or) obtain Laurent series exp of $f(z)=\frac{1}{(z^2+1)(z^2+2)}$ about $z=0$ (11)

Soln. Given $f(z) = \dfrac{1}{(z^2+1)(z^2+2)} = \dfrac{1}{(z^2+1)} - \dfrac{1}{(z^2+2)}$ ($\because$ By partial fractions)

Given region is $1<|z|<\sqrt{2}$

$$1<|z| \quad \& \quad |z|<\sqrt{2}$$

$$\frac{1}{|z|}<1 \quad \& \quad \frac{|z|}{\sqrt{2}}<1$$

$$\left|\frac{1}{z^2}\right|<1 \quad \& \quad \left|\frac{z^2}{2}\right|<1$$

$$\therefore f(z) = \frac{1}{(z^2+1)} - \frac{1}{(z^2+2)}$$

$$= \frac{1}{z^2\left(1+\frac{1}{z^2}\right)} - \frac{1}{2\left(1+\frac{z^2}{2}\right)}$$

$$= \frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1} - \frac{1}{2}\left(1+\frac{z^2}{2}\right)^{-1}$$

$$= \frac{1}{z^2}\left[1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right] - \frac{1}{2}\left[1-\frac{z^2}{2}+\frac{z^4}{2^2}-\frac{z^6}{2^3}+\cdots\right]$$

$$= \left(\frac{1}{z^2}-\frac{1}{z^4}+\frac{1}{z^6}-\frac{1}{z^8}+\cdots\right] - \left[\frac{1}{2}-\frac{z^2}{2^2}+\frac{z^4}{2^3}-\frac{z^6}{2^4}+\cdots\right]$$

$$= \left[\frac{1}{z^2}-\frac{1}{z^4}+\frac{1}{z^6}-\frac{1}{z^8}+\cdots\right] + \left[-\frac{1}{2}+\frac{z^2}{2^2}-\frac{z^4}{2^3}+\frac{z^6}{2^4}-\cdots\right]$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{2n+2}} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^{2n}}{2^{n+1}}$$

$$f(z) = \underbrace{\sum_{n=0}^{\infty}(-1)^n(z-0)^{-2n-2}}_{\text{negative power of } z} + \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+1}}(z-0)^{2n}}_{\text{+ve power of } z}$$

negative power of z · principal part of Laurent series

+ve power of z Analytic ~~part~~ part of Laurent series.

(11) Obtain all the Laurent series of the function

$$\frac{7z-2}{(z+1)\cdot z(z-2)} \text{ about } z=-1.$$

Soln. Given $f(z) = \dfrac{7z-2}{(z+1)\cdot z\cdot(z-2)}$

∴ The singular points of $f(z)$ are $z=0,-1,2$

∴ $f(z)$ is analytic everywhere except at $0,-1,2$.

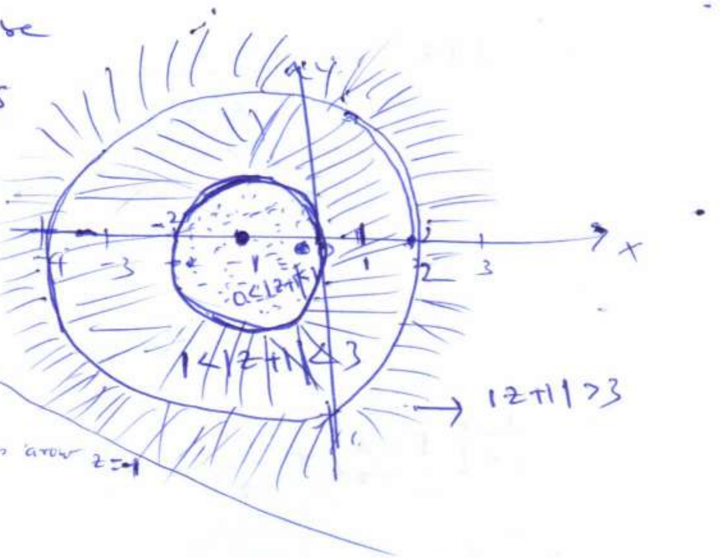
$$f(z) = \frac{7z-2}{z(z+1)(z-2)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-2} = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2}$$

(∵ partial fractions)

Three Laurent series of $f(z)$ can be obtained about $z=-1$ in the regions

(i) $0<|z+1|<1$ (ii) $1<|z+1|<3$

(iii) $|z+1|>3$



a) $0<|z+1|<1$ It is required to find Laurent series about $z=-1$

$$f(z) = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2}$$

$$= \frac{1}{(z+1)-1} - \frac{3}{z+1} + \frac{2}{(z+1)-3}$$

$\left|\frac{z+1}{3}\right|<1 \Rightarrow |z+1|<3$

$$= \frac{-1}{[1-(z+1)]} - \frac{3}{z+1} - \frac{2}{3\left[1-\frac{z+1}{3}\right]}$$

$$= (-1)[1-(z+1)]^{-1} - 3(z+1)^{-1} - \frac{2}{3}\left[1-\left(\frac{z+1}{3}\right)\right]^{-1}$$

$$= (-1)\left[1+(z+1)+(z+1)^2+(z+2)^3+\cdots\right] - 3(z+1)^{-1} - \frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2 + \left(\frac{z+1}{3}\right)^3+\cdots\right]$$

valid if $|z+1|<1$ & $z\neq-1$ & $\left|\frac{z+1}{3}\right|<1 \Rightarrow |z+1|<1$

$$f(z) = -3(z+1)^{-1} - \frac{5}{3} - \frac{11}{9}(z+1) - \frac{29}{27}(z+1)^2 - \frac{83}{81}(z+1)^3 - \cdots$$

b) for $1<|z+1|<3$

$1<|z+1|$ & $|z+1|<3$

$\dfrac{1}{|z+1|}<1$ & $\dfrac{|z+1|}{3}<1$

$$f(z) = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2}$$

$$f(z) = \frac{1}{(z+1)-1} - \frac{3}{(z+1)} + \frac{2}{(z+1)-3}$$

$$= \frac{1}{(z+1)\left[1-\frac{1}{z+1}\right]} - \frac{3}{(z+1)} + \frac{2}{3\left[1-\left(\frac{z+1}{3}\right)\right]}$$

$$= \frac{1}{(z+1)}\left[1-\frac{1}{z+1}\right]^{-1} - \frac{3}{z+1} - \frac{2}{3}\left[1-\left(\frac{z+1}{3}\right)\right]^{-1}$$

$$= \frac{1}{(z+1)}\left[1+\frac{1}{z+1}+\left(\frac{1}{z+1}\right)^2+\left(\frac{1}{z+1}\right)^3+\cdots\right] - \frac{3}{z+1} - \frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\right]$$

valid if $\left|\frac{1}{z+1}\right|<1$ & $z\neq -1$ & $\left|\frac{z+1}{3}\right|<1 \Rightarrow 1<|z+1|<3$.

$$= \left[\frac{1}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\cdots\right] - \frac{3}{z+1} - \frac{2}{3} - \frac{2}{9}(z+1) - \frac{2}{27}(z+1)^2 - \cdots$$

$$= \left[\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\cdots\right] - \frac{2}{z+1} - \frac{2}{3} - \frac{2}{9}(z+1) - \frac{2}{27}(z+1)^2 - \cdots$$

$$f(z) = \underbrace{\left[(z+1)^{-2}+(z+1)^{-3}+\cdots\right] - 2(z+1)^{-1}}_{-\text{ve power of } (z+1)} - \frac{2}{3} - \underbrace{\left[\frac{2}{9}(z+1)+\frac{2}{27}(z+1)^2+\cdots\right]}_{+\text{ve powers of } (z+1)}$$

c) For the Region $|z+1|>3 \Rightarrow |z+1|>3>1$

$\Rightarrow 3<|z+1|$ & $1<|z+1|$

$\Rightarrow \frac{3}{|z+1|}<1$ $\quad \frac{1}{|z+1|}<1$.

$$f(z) = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2}$$

$$= \frac{1}{(z+1)-1} - \frac{3}{z+1} + \frac{2}{(z+1)-3}$$

$$= \frac{1}{(z+1)\left[1-\frac{1}{z+1}\right]} - \frac{3}{z+1} + \frac{2}{(z+1)\left[1-\frac{3}{z+1}\right]}$$

$$= \frac{1}{(z+1)}\left[1-\frac{1}{z+1}\right]^{-1} - \frac{3}{z+1} + \frac{2}{(z+1)}\left[1-\frac{3}{z+1}\right]^{-1}$$

$$= \frac{1}{z+1}\left[1+\frac{1}{z+1}+\frac{1}{(z+1)^2}+\cdots\right] - \frac{3}{z+1} + \frac{2}{z+1}\left[1+\frac{3}{z+1}+\frac{9}{(z+1)^2}+\cdots\right]$$

$$= \left[\frac{1}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\cdots\right] - \frac{3}{z+1} + \left[\frac{2}{z+1}+\frac{6}{(z+1)^2}+\frac{18}{(z+1)^3}+\cdots\right]$$

$\therefore$ $f(z) = 7/(z+1)^2 + 19/(z+1)^3 + \cdots$ (only -ve power of $(z+1)$ is LS)

PART - B

[CONTOUR INTEGRATION]

We have studied the functions which are analytic in a given region. But there are several functions which are not analytic at certain points of its domain. Such exceptional points are called the "singularities" of the function & a type of a singular point is called a "pole". Now we study about different types of singularities & finding Residue of a function at a pole. also we prove residue theorem, which is useful to evaluate certain real integrals.

Definitions:

* **Zero (or) root of analytic function:** It is a value of z such that $f(z)=0$ (or) A point 'a' is called a zero of an analytic function $f(z)$ if $f(a)=0$.

  Ex. $f(z)=(z-1)$. Here $f(1)=0$ $\therefore$ '1' is called zero (or) root of $f(z)$.

* **Zero of m^th order:** Let $f(z)$ be analytic function, If the root 'a' of $f(z)$ repeated 'm' times then 'a' is called root (or) zero of m^th order & we write it as $f(z)=(z-a)^m\phi(z)$ where $\phi(a)\neq 0$.

  Ex. $f(z)=(z-1)^3$ $f(z)=(z-1)(z-1)(z-1)$

  $f(1)=0$

  then '1' is called zero of 3rd order.

  Ex. $f(z)=\dfrac{1}{1-z}$, then $z=\infty$ $f(\infty)=0$

  $\therefore \infty$ is called simple zero of $f(z)$

  Ex. $f(z)=\sin z$

  the simple zeros of $f(z)$ are $z=0, \pm\pi, \pm 2\pi, ---$

  Ex. $f(z)=e^{\tan z}$ has no zeros ($\because e^z\neq 0$ for any z.)

  ($f(a)\neq 0$ for any 'a')

  $\therefore f(z)$ has no zeros

Singular point: A singular point (or singularity) of a function $f(z)$ is the point at which the function $f(z)$ is not analytic.

(OR)

A point a, is said to be a singularity of $f(z)$, if $f(z)$ is not analytic at a.

Singularities are classified into two types

i) Isolated singularity. ii) Non-isolated singularity

Isolated singularity: A point $z=a$ is called an isolated singularity of an analytic function $f(z)$, if

i) $f(z)$ is not analytic at a

ii) $f(z)$ is analytic in the deleted nbd of $z=a$.

i.e. $f(z)$ is analytic in $\{z \,/\, 0<|z-a|<\delta\}$

Eg. $f(z) = \dfrac{1}{z-1}$

Here $z=1$ is a singularity of $f(z)$

further $z=1$ is a isolated singularity of $f(z)$ since $f(z)$ is analytic in the deleted nbd of $z=1$. [i.e. $\{0<|z-1|<\delta\}$]

Eg. $f(z) = \dfrac{1}{(z-1)(z-2)}$

Here $z=1,2$ are isolated singularities of $f(z)$

since $f(z)$ is analytic in the deleted nbd of

$0<|z-1|<\frac{1}{2}$, $0<|z-2|<\frac{1}{2}$

Eg: $f(z) = \dfrac{e^z}{z^2+1}$

$z=\pm i$ are two isolated singular points of $f(z)$

Eg. $f(z) = \dfrac{2}{\sin z}$

the isolated singular points are $z=\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ which are infinite in number.

Non-Isolated singularity: A singularity which is not isolated is called a non-isolated singularity.

i.e A singularity 'a' of $f(z)$ is said to be a non-isolated singularity if every nbd of 'a' contains a singularity other than 'a'.

Eg. $f(z) = \frac{1}{\sin(\frac{1}{z})}$

$\sin(\frac{1}{z}) = 0 \Rightarrow \frac{1}{z} = 0$ (or) $\frac{1}{z} = \pm n\pi \Rightarrow z = \frac{1}{n\pi}, n = \pm1, \pm2 \ldots$

The singularities of $f(z)$ are $\frac{1}{n\pi}, n = \pm1, \pm2, \ldots$

It may be noted that $\lim_{n\to\infty} \frac{1}{n\pi} = 0$

i.e $z=0$ is the limit of the seq of singularity

$\therefore$ Every nbd of '0' contain a singularity $\frac{1}{n\pi}$ for sufficiently large 'n'

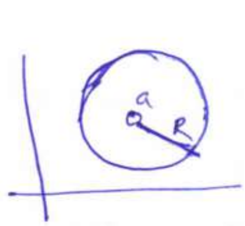
$\therefore z=0$ is a non-isolated singularity.

Classification of singularities

- singularity
  - Isolated singularity
    - Removable singularity
    - pole
    - Essential singularity.
  - Non-Isolated singularity.

→ If $z=a$ is an isolated singularity of $f(z)$, then $f(z)$ is analytic in a deleted nbd say, $0<|z-a|<R, R>0$

$\therefore f(z)$ has Laurent's Expansion which is valid in the Annulus $0<|z-a|<R$.



W.K.T the Laurent Series Exp of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \quad \text{valid in } 0<|z-a|<R$$

In this expansion $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called the Analytic part & $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ is called the principal part of the exp.

* Removable singularity: If the principle part of the Laurent exp of $f(z)$ around singular pt $z=a$ contains no terms, then singularity $z=a$ is said to be a "Removable singularity" of $f(z)$.

In this case $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$.

In this case the singularity can be removed by appropriately defining the function $f(z)$ at $z=a$ in such a way that it becomes analytic at $z=a$, such a singularity is called removable singularity.

Note: If $\lim_{z\to a} f(z) = \text{finite}$ then $z=a$ is a removable singularity

Ex: If $f(z) = \frac{1-\cos z}{z}$

then $z=0$ is a isolated singularity of $f(z)$ but

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{1-\cos z}{z} = \frac{0}{0} \text{ (Form)}$$

$$= \lim_{z\to 0} \frac{\sin z}{1} \quad \text{(L' Hospital Rule)}$$

$$= \underline{0} \text{ (finite)}$$

$\therefore z=0$ is called removable singularity of $f(z)$.

Ex: If $f(z) = \frac{\sin z}{z}$

$z=0$ is removable singularity.

(ii) poles: If the principle part of Laurent series exp of $f(z)$ around singular pt $z=a$ contains finite no of terms then, $z=a$ is called a pole.

If $b_m \neq 0$ & $b_k = 0$ for $k = m+1, m+2, \dots$

then $z=a$ is called a pole of order m.

A pole of order 1 is called a simple pole.

Eg: If $f(z) = \frac{z^2}{(z-1)(z+2)^2}$ $\quad z = 1, -2$ are isolated singular points.

Here $z=1$ is a simple pole.

& $z=-2$ is a pole of order

Essential singularity : If the principle part of the Laurent series exp of $f(z)$ around $z=a$ (singular pt), contains infinitely many terms then '$z=a$' is called an Essential singularity of $f(z)$.

Example for Removable singularity, pole & Essential Singularity.

Ex (1) Let $f(z) = \frac{\sin z}{z}$, $z \neq 0$ & it is isolated singularity

Here $z=0$ is a singularity of $f(z)$ & it is isolated singularity

Now $f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$ when $0<|z|<\infty$

$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ when $0<|z|<\infty$

$f(z) = 1 - \frac{(z-0)^2}{3!} + \frac{(z-0)^4}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(z-0)^{2n}}{\underbrace{(2n+1)!}_{A.P}}$

which is a Laurent Series Expansion of $f(z)$ about $z=0$. & it contains only positive powers (ie it contains only Analytic part & there is no principle part)

$\therefore$ $z=0$ is called removable singularity.

Redefining function $f(z)$

$f(z) = \begin{cases} 1 & \text{if } z=0 \\ \frac{\sin z}{z} & \text{if } z \neq 0 \end{cases}$

(OR) $f(z) = \frac{\sin z}{z}$, & $z=0$ is isolated singularity & $\lim_{z\to 0} f(z) = $ (finite value) $\therefore z=0$ is R.S.

Ex (2). Let $f(z) = \frac{z^2 - 2z + 3}{z-2} = \frac{z(z-2)+3}{z-2} = z + \frac{3}{z-2}$

$f(z) = z + \frac{3}{z-2}$

Here $z=2$ is a singular pt & it is isolated.

$f(z) = z + 3(z-2)^{-1}$ which is a Laurent Series exp of $f(z)$ around $z=2$ & it contains only one -ve power (ie it principle part contains only one term i.e finite no of terms).

$\therefore z=2$ is called a simple pole ~~of order 1~~.

Ex. Let $f(z) = e^{\frac{1}{z}}$

$$f(z) \equiv e^{\frac{1}{z}} = \frac{1}{e^{-\frac{1}{z}}}$$

the singular point are given by $e^{-\frac{1}{z}} = 0$

$$\Rightarrow \frac{1}{z} = \infty \qquad (\because e^{-\infty} = 0)$$

$$\Rightarrow \boxed{z=0}$$

$z=0$ is the singular point of $f(z)$ & it is isolated.

Now $f(z) = e^{\frac{1}{z}} = 1+\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots$ if $0<|z|<\infty$

$$= 1 + \frac{1}{1!(z-0)} + \frac{1}{2!}\cdot\frac{1}{(z-0)^2} + \frac{1}{3!}\frac{1}{(z-0)^3} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!\,(z-0)^n}$$

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}(z-0)^{-n}$$

which is a L.S. Exp of $f(z)$ about $z=0$ & it contains infinitely many ~~only~~ $-ve$ powers of $(z-0)$ [i.e principal part contains infinite no. of terms]

$\therefore z=0$ is called Essential singularity of $f(z)$.

(OR)

$\lim_{z\to 0} f(z) = e^{\frac{1}{z}}$ & $z=0$ is isolated singular pt

$\lim_{z\to 0} f(z)$ does not exist

$\therefore z=0$ is called essential singular pt.

Defi

* Singularity at infinity:

Let the function is $f(z)$

To find the singularity of $f(z)$ at $z=\infty$, then put $z=\frac{1}{t}$ in $f(z)$

then $f(z) = f\left(\frac{1}{t}\right) = F(t)$, [say]

Now ~~find~~ the singularity ~~points~~ of $F(t)$ at $t=0$ is the singularity of $f(z)$ at $z=\infty$.

Laurent's theorem: Let 'a' be an isolated singularity of an analytic function $f(z)$, then $f(z)$ can be expanded in ~~La~~ as $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n/(z-a)^n$ in $0<|z-a|<\delta$
which is called L.S. of $f(z)$ around $\underline{z=a}$

where $a_n = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz \;;\; n = 0, 1, 2, \ldots$

$$b_n = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^{-n+1}} dz \quad , n = 1, 2, 3 \ldots$$

where 'c' is truly oriented circle with centre at 'a', & lying entirely in the Annulus $0<|z-a|<\delta$.

<u>Residue at a pole:</u> : Let "$z=a$" be the pole of a function $f(z)$ then Residue of $f(z)$ at $z=a$ is denoted by $\operatorname{Res}_{z=a}[f(z)]$ & it is defined as the coefficient of $\frac{1}{z-a}$ in the Laurent series expansion ie b_1 is the Residue.

$$\text{ie } b_1 = \frac{1}{2\pi i} \int_c f(z)$$

$$\Rightarrow \int_c f(z) = 2\pi i \times b_1 = 2\pi i \times \operatorname{Res}_{\text{at } z=a}[f(z)]$$

→ If $z=a$ is the simple pole (pole order of 1) of $f(z)$ then

$$\operatorname{Res}_{z=a} f(z) = \lim_{z\to a} (z-a) f(z).$$

→ If $z=a$ is the ~~simple~~ pole of order 'm' of $f(z)$ then

$$\operatorname{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m \cdot f(z) \right]$$

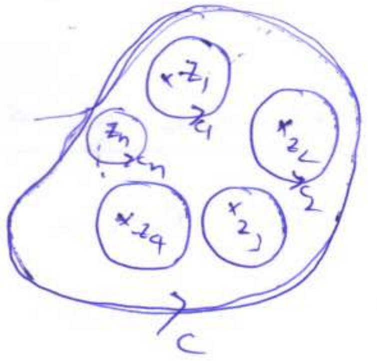
Cauchy's Residue Theorem:

Statement: Let 'c' be any +vely oriented simple closed contour. Let $f(z)$ is analytic on & within 'c', except at a finite number of poles $z_1, z_2, \dots z_n$ within 'c' and $R_1, R_2, \dots R_n$ be the residues of $f(z)$ at these poles, then $\int_c f(z)\,dz = 2\pi i\,[R_1 + R_2 + \dots + R_n]$

(OR)

$$\int_c f(z)\,dz = 2\pi i \times [\text{sum of the residues at the poles within } c]$$

Pf: Let $c_1, c_2, \dots c_n$ be the circles with centre at $z_1, z_2, \dots z_n$ respectively & the radii so small that all circle $c_1, c_2, \dots c_n$ are entirely lie in C & do not overlap.



Now $f(z)$ is analytic within the region enclosed by the curve 'c' between these circles {i.e.

∴ By Cauchy's theorem for multiply connected regions, we have

$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \dots + \int_{c_n} f(z)\,dz \quad (1)$$

But by definition we have.

$$\frac{1}{2\pi i}\int_{c_1} f(z)\,dz = \cancel{2\pi i}\,[\operatorname*{Res}_{at\ z=z_1} f(z)]$$

$\vdots$

$$\frac{1}{2\pi i}\int_{c_n} f(z)\,dz = \operatorname*{Res}_{at\ z=z_n} f(z)$$

Sub (1) ⇒

$$\int_c f(z)\,dz = 2\pi i \operatorname*{Res}_{at\ z=z_1} f(z) + 2\pi i \operatorname*{Res}_{at\ z=z_2} f(z) + \dots + 2\pi i \operatorname*{Res}_{at\ z=z_n} f(z)$$

$$= 2\pi i\left[\operatorname*{Res}_{at\ z=z_1} f(z) + \operatorname*{Res}_{at\ z=z_2} f(z) + \dots + \operatorname*{Res}_{at\ z=z_n} f(z)\right]$$

$$\int_c f(z)\,dz = 2\pi i\,[R_1 + R_2 + \dots + R_n] \quad //.$$

Problems Related to Poles & Residues.

Prob(1). Expand $f(z) = \frac{e^z}{(z-1)^2}$ as a Laurent series about $z=1$ & hence find the residue at that point.

Sol. Given $f(z) = \frac{e^z}{(z-1)^2}$ & $z=1$

It is Required to find L.S Exp around $z=1$ (ie in powers of $(z-1)$)

$$f(z) = (z-1)^{-2} e^{(z-1)+1} = (z-1)^{-2} \cdot e^{(z-1)} \cdot e$$

$$= e(z-1)^{-2}\left[1+(z-1)+\frac{(z-1)^2}{2!}+\frac{(z-1)^3}{3!}+\cdots\right]$$

$$= \frac{e}{(z-1)^2}\left[1+(z-1)+\frac{(z-1)^2}{2!}+\frac{(z-1)^3}{3!}+\cdots\right] \qquad (\because e^z = 1+z+z^2+\cdots)$$

$$= \frac{e}{(z-1)^2}+\frac{e}{z-1}+\frac{e}{2!}+\frac{e(z-1)}{9}+\cdots$$

$$f(z) = \underbrace{\left[\frac{e}{2}+\frac{e}{9}(z-1)+\cdots\right]}_{\text{+ve powers of } (z-1)} + \underbrace{\left[e(z-1)^{-1}+e(z-1)^{-2}+\cdots\right]}_{\text{-ve powers of } (z-1)}$$

+ve powers of $(z-1)$ — Analytical part. -ve powers of $(z-1)$ — principle part

The Given $f(z) = \frac{e^z}{(z-1)^2}$

The poles of $f(z)$ are given by $(z-1)^2 = 0 \Rightarrow z = 1, 1$

$\Rightarrow z=1$ is a pole of order 2

& Residue of $f(z)$ at $z=1$ is coefficient of $\frac{1}{z-1}$ in L.S. Exp.

ie $\underset{\text{at } z=1}{\text{Res}}\, f(z) = e$

(2) find the poles of the function i) $\frac{z}{\cos z}$ ii) $\cot z$ iii) $\frac{z}{z^2-3z+2}$

Sol. $f(z) = \frac{z}{\cos z}$

poles of $f(z)$ are given by Denominator $= 0$

i.e $\cos z = 0$

ie $z = (2n+1)\frac{\pi}{2}$; $n = 0, \pm 1, \pm 2 \ldots$

$\therefore$ The poles are : $z = \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}, \frac{5\pi}{2}, -\frac{3\pi}{2}, \ldots$ which are poles of order 1. [(or) simple poles]

(ii) $f(z) = \cot z$

$f(z) = \cot z = \frac{\cos z}{\sin z}$

poles are given by

$\sin z = 0$

ie $z = n\pi$ where $n = 0, \pm 1, \pm 2 \ldots$

$z = 0, \pi, -\pi, 2\pi, -2\pi, \ldots$ which ~~it~~ are called simple poles of $f(z)$

(iii) $f(z) = \frac{z^2}{z^2 - 3z + 2}$

poles are given by

$z^2 - 3z + 2 = 0$

$z^2 - 2z - z + 2 = 0$

$z(z-2) - 1(z-2) = 0$

$z = 1, 2$ ~~are~~ are called poles, which are simple poles (or poles of order 1)

(2) Find the poles of the function $f(z) = \frac{z^3}{(z-1)^4 (z-2)(z-3)}$ and residues at these poles.

Soln Given $f(z) = \frac{z^3}{(z-1)^4 (z-2)(z-3)}$

The poles of $f(z)$ are given by. $(z-1)^4 (z-2)(z-3) = 0$

$\Rightarrow$ $z = 1, 2, 3$

Here $z = 1$ is a pole of order 4, & $z = 2, 3$ are poles of order 1.

i) Residue at pole $z = 2$.

W.k.t if $z = a$ is a pole of order 1 then

W.K.T $\underset{\text{at } z=a}{\text{Res } f(z)} = \lim_{z \to a} (z-a) f(z)$

$\Rightarrow \underset{z=2}{\text{Res } f(z)} = \lim_{z \to 2} (z-2) \cdot f(z) = \lim_{z \to 2} \cancel{(z-2)} \cdot \frac{z^3}{(z-1)^4 \cancel{(z-2)} (z-3)}$

$= \frac{8}{1 \cdot (-1)} = -8$

(ii) Residue of $f(z)$ at $z=3$;

$$\underset{\text{at } z=3}{\text{Res } f(z)} = \underset{z\to 3}{\text{lt}} (z-3).f(z)$$

$$= \underset{z\to 3}{\text{lt}} \cancel{(z-3)}. \frac{z^3}{(z-1)^4.(z-2)\cancel{(z-3)}}$$

$$= \frac{27}{16.1.} = \frac{27}{16}$$

(iii) Residue of $f(z)$ at $z=1$

Here $z=1$ is a pole of order '4'

W.K.T If $z=a$ is a pole of order 'm' then

$$\underset{z=a}{\text{Res } f(z)} = \frac{1}{(m-1)!} \underset{z\to a}{\text{lt}} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

Here $m=4$, $a=1$

$$\Rightarrow \underset{\text{at } z=1}{\text{Res } f(z)} = \frac{1}{3!} \underset{z\to 1}{\text{lt}} \frac{d^3}{dz^3}\left[\cancel{(z-1)^4}. \frac{z^3}{\cancel{(z-1)^4}.(z-2)(z-3)}\right]$$

$$\underset{\text{at } z=1}{\text{Res } f(z)} = \frac{1}{6}.\underset{z\to 1}{\text{lt}} \frac{d^3}{dz^3}\left[\frac{z^3}{(z-2)(z-3)}\right] \quad \text{—— (i)}$$

Let us find out $\frac{d^3}{dz^3}\left[\frac{z^3}{(z-2)(z-3)}\right]$

Now $\frac{z^3}{(z-2)(z-3)} = Az + B + \frac{C}{z-2} + \frac{D}{z-3}$ (∵ By partial fraction)

Here $A=1$, $B=5$, $C=-8$, $D=27$ (find out)

$$\therefore \frac{z^3}{(z-2)(z-3)} = z+5 - \frac{8}{(z-2)} + \frac{27}{(z-3)}$$

$$\frac{d^3}{dz^3}\left[\frac{z^3}{(z-2)(z-3)}\right] = \frac{d^3}{dz^3}\left[z+5-\frac{8}{z-2}+\frac{27}{z-3}\right]$$

$$= \frac{d^2}{dz^2}\left[\frac{d}{dz}\left\{z+5-\frac{8}{z-2}+\frac{27}{z-3}\right\}\right]$$

$$= \frac{d^2}{dz^2}\left[1+0+\frac{8}{(z-2)^2}-\frac{27}{(z-3)^2}\right]$$

$$= \frac{d}{dz}\left[\frac{d}{dz}\left\{1+0+\frac{8}{(z-2)^2}-\frac{27}{(z-3)^2}\right\}\right] = \frac{d}{dz}\left[\frac{-16}{(z-2)^3}+\frac{54}{(z-3)^3}\right] = \frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}$$

$$\therefore \frac{d^3}{dz^3}\left[\frac{z^3}{(z-2)(z-3)}\right] = \frac{48}{(z-2)^4} - \frac{162}{(z-3)^4} \quad \text{---} (2)$$

Sub (2) in (1)

$$\underset{z=1}{\text{Res}}\, f(z) = \frac{1}{6} \underset{z\to 1}{\text{Lt}} \left[\frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}\right]$$

$$= \frac{1}{6}\left[\frac{48}{1} - \frac{162}{16}\right]$$

$$= \frac{1}{6}\left[\frac{303}{8}\right] = \frac{101}{16}$$

$$\therefore \underset{\text{at } z=1}{\text{Res}\, f(z)} = \frac{101}{16}$$

(3) find the residue at $z=0$ of the function $f(z) = \frac{1+e^z}{\sin z + z\cos z}$.

Soln. Given $f(z) = \frac{1+e^z}{\sin z + z\cos z}$

Given pole of $f(z)$ is $z=0$. (~~simple pole~~ (or) pole of order 1)

<u>Residue of $f(z)$ at $z=0$</u>: since $z=0$ is a pole of order '1'

$$\therefore \underset{\text{at } z=0}{\text{Res}\, f(z)} = \underset{z\to 0}{\text{lt}} (z-0)\, f(z)$$

$$= \underset{z\to 0}{\text{lt}}\, z \cdot \left[\frac{1+e^z}{\sin z + z\cos z}\right]$$

$$= \underset{z\to 0}{\text{lt}} \frac{\cancel{z}(1+e^z)}{\cancel{z}\left(\frac{\sin z}{z} + \cos z\right)} = \frac{\underset{z\to 0}{\text{lt}}(1+e^z)}{\underset{z\to 0}{\text{lt}}\left(\frac{\sin z}{z} + \cos z\right)}$$

$$= \frac{1+e^0}{1+\cos 0} = \frac{2}{2} = 1$$

$$\therefore \underset{\text{at } z=0}{\text{Res}\,[f(z)]} = 1$$

(4) Find the residue of the function $f(z) = \frac{1-e^{2z}}{z^4}$, at the poles.

Soln: $z=0$ is the singular point. (21)

L.S. Exp of $f(z)$ around $z=0$ (or in powers of $(z-0)$)

$$f(z) = \frac{1-e^{2z}}{z^4} = \frac{1}{z^4}\left[1-\left(1+2z+\frac{(2z)^2}{2!}+\frac{(2z)^3}{3!}+\frac{(2z)^4}{4!}+\cdots\right)\right]$$

$$= \frac{1}{z^4}\left[-2z-\frac{4z^2}{2}-\frac{8z^3}{3!}-\frac{16z^4}{24}-\cdots\right]$$

$$f(z) = -\left[\frac{2}{z^3}+\frac{2}{z^2}+\frac{4}{3}\cdot\frac{1}{z}+\frac{2}{3}+\frac{4}{15}z+\cdots\right]$$

$f(z) = -\left[2(z-0)^{-3}+2(z-0)^{-2}+\frac{4}{3}(z-0)^{-1}+\frac{2}{3}+\frac{4}{15}(z-0)+\cdots\right]$ which is L.S Exp of $f(z)$ in powers of $(z-0)$

$z=0$ is a pole of order 3, bcz $\frac{1}{(z-0)^3}$ is the highest -ve power $(z-0)$

Residue of $f(z)$ at $z=0$ is the coefficient of $\frac{1}{z}$ i.e $-\frac{4}{3}$

$$\therefore \operatorname*{Res}_{\text{at } z=0} f(z) = -4/3$$

(5) Determine the poles of the function $f(z)$ where $f(z) = \dfrac{e^z}{z^2+\pi^2}$ and the residues at these poles.

Soln. Given $f(z) = \dfrac{e^z}{z^2+\pi^2}$

The poles of $f(z)$ are given by $z^2+\pi^2=0$

$$\Rightarrow z^2 = -\pi^2 = i^2\pi^2$$

$$\Rightarrow z = \pm i\pi$$

$z = \pi i, -\pi i$ are the poles of order 1

Residue at $z=\pi i$

$$\operatorname*{Res}_{\text{at } z=\pi i} f(z) = \lim_{z\to\pi i}(z-\pi i)\cdot f(z) = \lim_{z\to\pi i}(z-\pi i)\cdot\frac{e^z}{z^2+\pi^2}$$

$$= \lim_{z\to\pi i}\cancel{(z-\pi i)}\cdot\frac{e^z}{(z+\pi i)\cancel{(z-\pi i)}}$$

$$= \frac{e^{\pi i}}{\pi i+\pi i} = \frac{e^{\pi i}}{2\pi i} = \frac{\cos\pi + i\sin\pi}{2\pi i}$$

$$= \frac{-1}{2\pi i}$$

Residue at $z=-\pi i$

$$\operatorname*{Res}_{\text{at } z=-\pi i} f(z) = \frac{1}{2\pi i}$$

(6) find the residue of $f(z) = \frac{z^3}{z^2-1}$ at $z=\infty$

Soln. w.k.t $\underset{\text{at } z=\infty}{\text{Res } f(z)} = \cancel{\underset{\text{at } z=0}{\text{Res } f(1/z)}}$ - coefficient of $\frac{1}{z}$ in L.S Exp of $f(z)$

$$f(z) = \frac{z^3}{z^2-1} = \frac{z^3}{z^2(1-\frac{1}{z^2})}$$

$$= z\left[1-\frac{1}{z^2}\right]^{-1}$$

$$= z\left[1+\frac{1}{z^2}+\frac{1}{z^4}+\frac{1}{z^6}+\cdots\right]$$

$$f(z) = z+\frac{1}{z}+\frac{1}{z^3}+\frac{1}{z^5}+\cdots$$ which is L.S Exp of $f(z)$

$$\therefore \underset{\text{at } z=\infty}{\text{Res } f(z)} = -\left[\text{coefficient of } \frac{1}{z}\right]$$

$$= -1$$

(7) find the residue of $\frac{z^2}{z^4+1}$ at those poles which lie inside the circle $|z|=2$

Soln. Let $f(z) = \frac{z^2}{z^4+1}$

Poles are given by $z^4+1=0$

$$z^4=-1$$

$$z=(-1)^{1/4}$$

$$=(\cos\pi+i\sin\pi)^{1/4} \qquad (\because \cos\pi+i\sin\pi=-1)$$

$$=\left[\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)\right]^{1/4}$$

$$z = \cos\left(\frac{2n\pi+\pi}{4}\right)+i\sin\left(\frac{2n\pi+\pi}{4}\right) \qquad \left[\because (\cos\theta+i\sin\theta)^n = \cos n\theta+i\sin n\theta\right]$$

when $n=0,1,2,3$

$\therefore$ the four values of 'z' are.

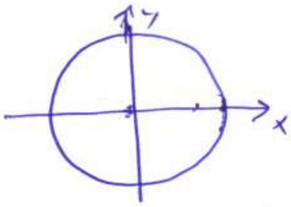
$\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$, $\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}$, $\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}$, $\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}$

$\Rightarrow \frac{1}{\sqrt2}+\frac{i}{\sqrt2}$, $\frac{-1}{\sqrt2}+\frac{i}{\sqrt2}$, $\frac{-1}{\sqrt2}-\frac{i}{\sqrt2}$, $\frac{1}{\sqrt2}-\frac{i}{\sqrt2}$

$\Rightarrow \frac{1+i}{\sqrt2}$, $-1+i/\sqrt2$, $\frac{-1-i}{\sqrt2}$ & $\frac{1-i}{\sqrt2}$ $\Rightarrow (\frac{1}{\sqrt2},\frac{1}{\sqrt2})(\frac{-1}{\sqrt2},\frac{1}{\sqrt2})(\frac{-1}{\sqrt2},\frac{-1}{\sqrt2})(\frac{1}{\sqrt2},\frac{-1}{\sqrt2})$

which are poles of order '1'

All the four poles of $f(z)$ lie inside the circle $|z|=2$



$|z|=2$ is a circle with centre $(0,0)$ & $r=2$

Residue of $f(z)$ at $z=\frac{1+i}{\sqrt{2}}$

w.k.t $\underset{\text{at } z=a}{\operatorname{Res} f(z)} = \underset{z\to a}{Lt}\,(z-a)\cdot f(z)$

$$\Rightarrow \underset{\text{at } z=\frac{1+i}{\sqrt{2}}}{\operatorname{Res} f(z)} = \underset{z\to\frac{1+i}{\sqrt{2}}}{Lt}\left(z-\left(\frac{1+i}{2}\right)\right)\cdot\frac{z^2}{(z^4+1)}$$

$$= \underset{z\to\frac{1+i}{\sqrt{2}}}{Lt}\left[\cancel{z-\left(\frac{1+i}{\sqrt{2}}\right)}\right]\cdot\frac{z^2}{\cancel{\left[z-\left(\frac{1+i}{\sqrt{2}}\right)\right]}\left[z-\left(\frac{-1+i}{\sqrt{2}}\right)\right]\left[z-\left(\frac{-1-i}{\sqrt{2}}\right)\right]\left[z-\left(\frac{1-i}{\sqrt{2}}\right)\right]}$$

$$= \frac{\left(\frac{1+i}{\sqrt{2}}\right)^2}{(\quad)(\quad)(\quad)} = \frac{1-i}{4\sqrt{2}}$$

||ly we find the Residues at other poles

(8) find the poles & residues at each pole of $f(z)=\frac{1}{(z^2+4)^2}$

Soln. $f(z) = \frac{1}{(z^2+4)^2}$

The poles of $f(z)$ are given by $(z^2+4)^2=0$

$$\Rightarrow (z^2+4)(z^2+4)=0 \quad \Rightarrow z^2+4=0 \text{ (or) } z^2+4=0$$

$$\Rightarrow z^2=-4 \text{ (or) } z^2=-4$$

$$\Rightarrow z^2=4i^2 \text{ (or) } z^2=4i^2$$

$$\Rightarrow z=\pm\sqrt{4i^2} \text{ (or) } z=\pm\sqrt{4i^2}$$

$$\Rightarrow z=\pm 2i,\ \pm 2i$$

$z = 2i, 2i, -2i, -2i$

$z=2i$ which is a pole of order 2.

$z=-2i$ which is a pole of order 2.

i) Residue of $f(z)$ at $z=2i$:

$$\underset{\text{at } z=2i}{\operatorname{Res} f(z)} = \frac{1}{1!}\underset{z\to 2i}{Lt}\frac{d}{dz}\left[(z-2i)^2\cdot f(z)\right]$$

$$= \underset{z\to 2i}{Lt}\left[(z-2i)\right]^2\cdot\frac{1}{(z^2+4)^2}$$

$$= \lim_{z \to 2i} \left[\cancel{(z-2i)}\right]^2 \frac{1}{\cancel{(z-2i)(z-2i)}(z+2i)(z+2i)}$$

$\text{Res } f(z)$ at $z = 2i$ $= -\dfrac{i}{16}$

Similarly find residues at all the ~~poles~~ remaining poles.

(9) find the poles & residues at each pole of $\tanh z$

Soln. Let $f(z) = \tanh z = \dfrac{\sinh z}{\cosh z} = \dfrac{\frac{e^z - e^{-z}}{2}}{\frac{e^z + e^{-z}}{2}} = \dfrac{e^z - e^{-z}}{e^z + e^{-z}}$

$$= \frac{e^z + \frac{1}{e^z}}{e^z + \frac{1}{e^z}} = \frac{e^{2z}-1}{e^{2z}+1}$$

$$\therefore f(z) = \frac{e^{2z}-1}{e^{2z}+1}$$

poles of $f(z)$ are given by $e^{2z}+1 = 0$.

$$\Rightarrow (e^z)^2 + (1)^2 = 0$$

$$\Rightarrow (e^z)^2 = -1 \qquad \Rightarrow (e^z)^2 = i^2$$

$$\Rightarrow e^z = \pm\sqrt{i^2}$$

$\Rightarrow e^z = i, -i$

$\Rightarrow e^z = i$, & $e^z = -i$

$\Rightarrow e^z = e^{\frac{i\pi}{2}}$ & $e^z = e^{-\frac{i\pi}{2}}$

$\Rightarrow z = \frac{i\pi}{2}, -\frac{i\pi}{2}$ which are poles of order 1.

(i) Res of $f(z)$ at $z = \frac{\pi i}{2}$

$$\text{Res } f(z) \text{ at } z = \frac{\pi i}{2} = \lim_{z \to \frac{\pi i}{2}} \left(z - \frac{\pi i}{2}\right) \cdot f(z)$$

$$= \lim_{z \to \frac{\pi i}{2}} \cdot \left(z - \frac{\pi i}{2}\right) \cdot \frac{e^{2z}-1}{(e^{2z}+1)}$$

$$= \lim_{z \to \frac{\pi i}{2}} \cancel{\left(z - \frac{\pi i}{2}\right)} \frac{e^{2z}-1}{\cancel{\left(z - \frac{\pi i}{2}\right)}\left(z + \frac{\pi i}{2}\right)}$$

$$= \frac{e^{\pi i}-1}{\frac{\pi i}{2} + \frac{\pi i}{2}} = \frac{e^{\pi i}-1}{\pi i} = \frac{-2}{\pi i} = \frac{2i}{\pi}$$

(ii)

(10) find the poles & residues at each pole of $\frac{\cot z \coth z}{z^3}$

Sol: Let $f(z) = \frac{\cot z \coth z}{z^3}$

the poles of $f(z)$ are given by $z^3 = 0$

$\Rightarrow z = 0$ is a pole of order '3'

$$f(z) = \frac{\cot z \coth z}{z^3} = \frac{\cos z \cosh z}{z^3 \sin z \sinh z}.$$

$$= \frac{1}{z^3} \cdot \frac{\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right]\left[1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right]}{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]}$$

$$= \frac{1}{z^3} \cdot \frac{1 + \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^2}{2!} - \frac{z^4}{(2!)^2} - \frac{z^6}{2!4!} + \frac{z^4}{4!} + \frac{z^6}{4!2!} + \cdots}{z^2 + \frac{z^4}{3!} + \frac{z^6}{5!} - \frac{z^4}{3!} - \frac{z^6}{(3!)^2} - \frac{z^8}{3!5!} + \frac{z^6}{5!} + \frac{z^8}{5!3!} + \cdots}$$

$$= \frac{1}{z^3} \cdot \frac{\cancel{1+} z^4\left(\frac{1}{12} - \frac{1}{4}\right) + \cdots}{z^2 + z^6\left[\frac{2}{5!} - \frac{1}{(3!)^2}\right] + \cdots}$$

$$= \frac{1}{z^3} \cdot \frac{1 - \frac{1}{6}z^4 + \cdots}{z^2 - \frac{1}{90}z^6 + \cdots}$$

$$= \frac{1 - \frac{1}{6}z^4 + \cdots}{z^5\left[1 - \frac{1}{90}z^4 + \cdots\right]}$$

$$= \frac{1}{z^5}\left[1 - \frac{1}{6}z^4 + \cdots\right]\left[1 - \frac{1}{90}z^4 + \cdots\right]^{-1}$$

$$= \frac{1}{z^5}\left[1 - \frac{1}{6}z^4 + \cdots\right]\left[1 + \frac{1}{90}z^4 - \cdots\right]$$

$$= \frac{1}{z^5}\left[1 + \frac{1}{90}z^4 - \frac{1}{6}z^4 + \cdots\right]$$

$$= \frac{1}{z^5}\left[1 - \frac{14}{90}z^4 + \cdots\right]$$

$f(z) = \frac{1}{z^5} - \frac{7}{45}\frac{1}{z} + \cdots$ which is L.S Exp of $f(z)$ about '0'

$\operatorname{Res}_{at\ z=0} f(z) =$ the coefficient of $\frac{1}{z} = -\frac{7}{45}$ //

(11) find the Residues of $f(z) = \dfrac{1}{z(e^z-1)}$

Soln. Given $f(z) = \dfrac{1}{z(e^z-1)}$

The poles of $f(z)$ are given by $z(e^z-1)=0$

$\Rightarrow \quad z=0 \quad (or) \quad e^z-1=0$

$\Rightarrow e^z = 1$

$\Rightarrow e^z = e^{2n\pi i}, n=0, \pm1, \pm2, \dots$

$\Rightarrow z = 2n\pi i$

$\therefore$ The poles are $z=0,\ 2n\pi i.\ n=0, \pm1, \pm2, \dots$

when $n=0$ then $z=0.0$

$\therefore z=0$ is a pole of order 2.

$$f(z) = \frac{1}{z(e^z-1)} = \frac{1}{z\left[\left(\cancel{1}+z+\frac{z^2}{2}+\frac{z^3}{3!}+\dots\right)-\cancel{1}\right]}$$

$$= \frac{1}{z\times z\left[1+\frac{z}{2}+\frac{z^2}{3!}+\dots\right]}$$

$$= \frac{1}{z^2}\left[1+\left(\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)\right]^{-1}$$

$$= \frac{1}{z^2}\left[1-\left(\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)+\left(\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)^2-\dots\right]$$

$$= \frac{1}{z^2}\left[1-\frac{z}{2!}+\left(\frac{1}{4}-\frac{1}{6}\right)z^2+\left(-\frac{1}{24}+\frac{1}{6}-\frac{1}{8}\right)z^3+\dots\right]$$

$$= \frac{1}{z^2}\left[1-\frac{z}{2}+\frac{1}{12}z^2+\frac{1}{360}z^4+\dots\right]$$

$$f(z) = \frac{1}{z^2}-\frac{1}{2z}+\frac{1}{12}+\frac{1}{360}z^2+\dots$$

$$f(z) = \frac{1}{(z-0)^2}-\frac{1}{2(z-0)}+\frac{1}{12}+\frac{1}{310}(z-0)^2+\dots$$ which is a L.S Exp of $f(z)$ in powers of $(z-0)$ (or) about pole $\underline{z=0}$.

$\therefore \underset{at\ z=0}{\text{Res}\ f(z)} = \text{coefficient of } \frac{1}{z} = -\frac{1}{2}$ //.

H/W :1) find the poles & the corresponding residues of $f(z) = \dfrac{e^z}{(1+z)^2}$

2) $\dfrac{z^2}{z^4-1}$ Hint: $z^4-(1)^2 = (z^2)^2-(1)^2 = (z^2-1)(z^2+1) \Rightarrow z=\pm1,\ z=\pm i$

3) $\dfrac{ze^{zt}}{(z-3)^2}$ (4) $f(z) = \dfrac{z^2+2z}{(z+1)^2(z^2+4)}$ (5) $\dfrac{ze^t}{(z-1)^3}$

(6) $f(z) = \dfrac{z^2}{(z+1)^2(z+2)}$ (7) $f(z) = \dfrac{\cos(z-i)}{(z+2i)^3}$

Problems Related to Evaluation of Integrals Using Residue theorem

Prob ①. Evaluate $\oint_C \frac{4-3z}{z(z-1)(z-2)}\,dz$ where 'c' is the circle $|z| = \frac{3}{2}$ using Residue theorem.

soln : Let $f(z) = \frac{4-3z}{z(z-1)(z-2)}$.

The poles of $f(z)$ are given by $z(z-1)(z-2) = 0$

$\Rightarrow z = 0, 1, 2$

$z = 0, 1, 2$ are the poles of order '1'

The given curve 'c' is $|z| = \frac{3}{2}$ which is a circle

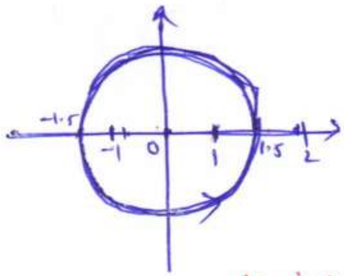
$\Rightarrow |z-0| = \frac{3}{2}$

$\Rightarrow |x+iy-0| = \frac{3}{2}$

$\Rightarrow |(x-0)+iy| = \frac{3}{2}$

$\sqrt{(x-0)^2+y^2} = \frac{3}{2}$

$\Rightarrow (x-0)^2 + (y-0)^2 = 1.5$



$|x+iy| = \sqrt{x^2+y^2}$

which is a circle with centre (0,0) & r = 1.5

The poles z = 0, 1 are only lies inside the curve 'c'

∴ we Required to find the Residues at the poles z = 0, 1

Residue of f(z) at z = 0 :

w.k.t $\underset{\text{at } z=a}{\text{Res}\ f(z)} = \lim_{z \to a} (z-a) \cdot f(z)$ | if z = a is a simple pole

$R_1 = \underset{\text{at } z=0}{\text{Res}\ f(z)} = \lim_{z \to 0} \cancel{(z-0)} \cdot \frac{4-3z}{\cancel{z}(z-1)(z-2)}$

$= \frac{4}{2} = 2$ $\quad \therefore \boxed{R_1 = 2}$

Residue of f(z) at z = 1 ,

$R_2 = \underset{\text{at } z=1}{\text{Res}\ f(z)} = \lim_{z \to 1} (z-1) \cdot f(z)$

$= \lim_{z \to 1} \cancel{(z-1)} \cdot \frac{4-3z}{z\cancel{(z-1)}(z-2)}$

$= \frac{1}{1(-1)} = -1$ $\quad \therefore \boxed{R_2 = -1}$

∴ By Cauchy-Residue theorem.

$\oint_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\pi i\,(R_1 + R_2)$
$= 2\pi i\,(2-1) = 2\pi i$ //

| $\oint_C f(z)\,dz = 2\pi i$ (sum of Residues) $= 2\pi i(R_1 + R_2 + \dots R_n)$

② Obtain L.S for the function $f(z) = \frac{1}{z^2 \sinh z}$ & evaluate

$\int_c \frac{dz}{z^2 \sinh z}$: where 'c' is the circle $|z-1| = 2$

Soln: Given $f(z) = \frac{1}{z^2 \sinh z}$

$$= \frac{1}{z^2\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]} \qquad \left(\because \sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)$$

$$= \frac{1}{z^3\left[1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right]}$$

$$= \frac{1}{z^3}\left[1 + \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)\right]^{-1}$$

$$= \frac{1}{z^3}\left[1 - \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right) + \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)^2 - \cdots\right]$$

$$= \frac{1}{z^3}\left[1 - \frac{z^2}{6} - \frac{z^4}{120} + \frac{z^4}{36} + \cdots\right] \qquad \left(\because (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\right)$$

$$= \frac{1}{z^3}\left[1 - \frac{z^2}{6} + \left(\frac{1}{36} - \frac{1}{120}\right)z^4 - \cdots\right]$$

$f(z) = \frac{1}{z^3} - \frac{1}{6z} + \frac{7}{360}z^4 - \cdots$ which is called L.S exp of $f(z)$ about '0' (in power of $z-0$)

The highest -ve power of $(z-0)$ is '3'

$\therefore z = 0$ is a pole of order '3'

The pole $z=0$ lies inside 'c'

The given curve 'c' is $|z-1| = 2$

$|(x+iy)-1| = 2$

$|(x-1)+iy| = 2$

$\sqrt{(x-1)^2+y^2} = 2$

$(x-1)^2+(y-0)^2 = 2$

$(1,0)$ $r=2$

$R_1 = \underset{\text{at } z=0}{\text{Res } f(z)}$ = coefficient of $\frac{1}{z}$ in L.S exp $= -\frac{1}{6}$

The given curve

By Residue theorem,

$$\int_c f(z)\,dz = 2\pi i\,[\text{sum of residues}]$$

$$\Rightarrow \int_c \frac{1}{z^2 \sinh z}\,dz = 2\pi i\,[R_1] = 2\pi i \times \frac{-1}{6}$$

$$= \frac{-\pi i}{3} \quad //$$

(3) Evaluate $\oint_C \frac{z-3}{z^2+2z+5}\,dz$, where 'C' is the circle given by i) $|z+1-i|=2$ ii) $|z|=1$

Soln. Given $f(z) = \frac{z-3}{z^2+2z+5}$

The poles of $f(z)$ are given by $z^2+2z+5=0$

$\Rightarrow$ $z = \frac{-2 \pm \sqrt{4-4\cdot1\cdot5}}{2\cdot1}$

$\Rightarrow$ $z = \frac{-2 \pm \sqrt{-16}}{2}$

$= \frac{-2 \pm \sqrt{i^2\cdot 16}}{2}$

$z = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

$\Rightarrow z = -1+2i$, $z = -1-2i$ [ie $z = (-1,2)$ & $(-1,-2)$]

which are poles of order '1'

now the given curve 'c' is $|z+1-i|=2$

$\Rightarrow |x+iy+1-i|=2$

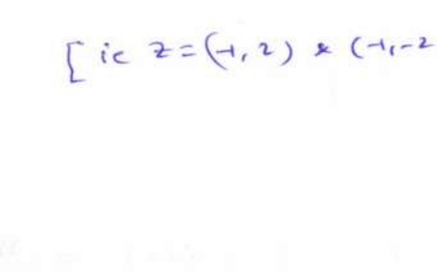
$\Rightarrow |(x+1)+i(y-1)|=2$

$\Rightarrow \sqrt{(x+1)^2+(y-1)^2}=2$

$\Rightarrow (x+1)^2+(y-1)^2=2$ which is a circle with centre $(-1,1)$ & $r=2$

Here only the pole $-1-2i$ [ie $(-1,-2)$] lies inside 'c'.

$\therefore$ $f(z)$ is analytic every where in 'c' Except at $z=-1-2i$



Residue of $f(z)$ at $z=-1-2i$

$R_1 = \underset{\text{at } z=-1-2i}{\text{Res } f(z)} = \lim_{z\to(-1-2i)} [z-(-1-2i)] \frac{z-2}{z^2+2z+5}$

$\underset{z=a}{\text{Res}\{f(z)\}} = \lim_{z\to a}(z-a)f(z)$

$= \lim_{z\to(-1-2i)} \cancel{[z-(-1-2i)]} \times \frac{z-3}{[z-(-1+2i)]\cancel{[z-(-1-2i)]}}$

$= \lim_{z\to(-1-2i)} \frac{z-3}{z+1-2i} = \frac{-1-2i-3}{\cancel{-1}-2i\cancel{+1}-2i} = \frac{-4-2i}{-4i}$

$= \frac{2+i}{2i} = -\frac{i}{i}+\frac{1}{2}$

Hence $\oint_C \frac{z-3}{z^2+2z+5}dz = 2\pi i[R_1] = 2\pi i\left(-i+\frac{1}{2}\right) = \pi(2+i)$

ii) $C: |z|=1$

Both the poles lies out side the circle.

$\therefore$ $f(z)$ is analytic every where inside 'c'

$\therefore$ By Cauchy's thm

$\oint_C f(z)\,dz = 0$

$\therefore \oint_C \frac{z-3}{z^2+2z+5}dz = 0$

(4) Evaluate $\int_C \frac{dz}{\sinh z}$ where 'c' is the circle $|z|=4$ using Residue theorem.

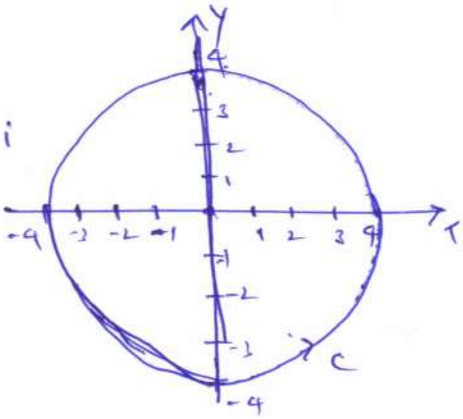
Sol: Given $f(z) = \frac{1}{\sinh z}$

the poles of $f(z)$ are given by $\sinh z = 0$

$\Rightarrow z = \pm n\pi i, \; n = 0, \pm 1, \pm 2 \ldots$

$\Rightarrow z = 0, \pi i, -\pi i, 2\pi i, -2\pi i \ldots$ which are poles of order 1. $[(0,0) \; (0,\pi) \; (0,-\pi) \; (0,2\pi), (0,-2\pi) \ldots]$

The given curve 'c' is $|z| = 4$ which is a circle with centre $(0,0)$ & radius $(r) = 4$.



Here the only poles lies inside curve 'c' are $z = 0, \pi i, -\pi i$

Residue at $z=0$:

$$R_1 = \underset{\text{at } z=0}{\text{Res } f(z)} = \lim_{z\to 0} (z-0)\cdot f(z)$$

$$= \lim_{z\to 0} z \cdot \frac{1}{\sinh z}$$

$$= \frac{0}{0} \quad (\text{Indeterminate form})$$

$$= \lim_{z\to 0} \frac{1}{\cosh z} \quad (\text{L' Hospital Rule})$$

$$= \frac{1}{\cos 0} = \frac{1}{1}$$

$\therefore \boxed{R_1 = 1}$

Residue at $z = \pi i$:

$$R_2 = \underset{\text{at } z=\pi i}{\text{Res } f(z)} = \lim_{z\to \pi i} (z-\pi i)\cdot f(z)$$

$$= \lim_{z\to \pi i} (z-\pi i)\cdot \frac{1}{\sinh z}$$

$$= \frac{(\pi i - \pi i)}{\sinh(\pi i)}$$

$$= \frac{0}{0} \; (\text{Ind form})$$

$$= \lim_{z\to\pi i} \frac{1}{\cosh z} \quad (\text{L' Hospital Rule})$$

$$= \frac{1}{\cosh(\pi i)}$$

$$= \frac{1}{-1} = -1$$

$\sinh \pi i = \frac{e^{\pi i} - e^{-\pi i}}{2} = 0$

$\cosh \pi i = \frac{e^{\pi i} + e^{-\pi i}}{2} = -1$

$\therefore \boxed{R_2 = -1}$

Residue at $z = -\pi i$: $\boxed{R_3 = -1}$

Hence $\int$

hence by cauchy's Residue theorem. (26)

$$\oint_C f(z)\,dz = 2\pi i\,[\text{sum of the Residues}]$$

$$\Rightarrow \oint_C \frac{1}{\sinh z}\,dz = 2\pi i\,[1-1-1] = \underline{-2\pi i}$$

(5) Evaluate $\int_C \frac{dz}{z\sin z}$ where 'c' is the unit circle with centre at the origin.

Soln. Given $f(z) = \frac{1}{z\sin z}$

the poles of $f(z)$ are given by $z\sin z = 0$

$\Rightarrow z = 0$ (or) $\sin z = 0$

$\Rightarrow z = n\pi \;;\; n = 0, \pm 1, \pm 2, \cdots$

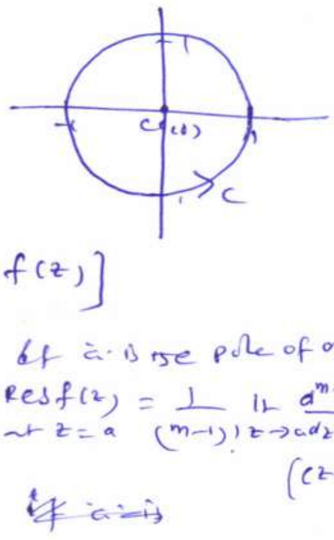
$\Rightarrow z = 0, \pi, -\pi, 2\pi, -2\pi, \cdots$

the poles are $z = 0, 0, \pi, -\pi, 2\pi, -2\pi, \cdots$ ~~which are poles of order 1~~

$z=0$ is pole of order '2' & remaining all poles are of order '1'

Given curve 'c' is the circle with centre $(0,0)$ & $r=1$

Here '$z=0$' is the only pole which lies inside 'c'.



<u>Residue at $z=0$</u>: since $z=0$ is a pole of order 2

$$\therefore R_1 = \underset{\text{at } z=0}{\operatorname{Res}} f(z) = \frac{1}{(2-1)!}\lim_{z\to 0}\frac{d^{2-1}}{dz^{2-1}}\left[(z-0)^2\cdot f(z)\right]$$

If a is the pole of order m then
$\underset{\text{at } z=a}{\operatorname{Res}} f(z) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$

$$= \lim_{z\to 0}\frac{d}{dz}\left[z^{\cancel{2}}\cdot\frac{1}{\cancel{z}\sin z}\right]$$

$$= \lim_{z\to 0}\left[\frac{1\cdot\sin z - z\cdot\cos z}{\sin^2 z}\right]$$

$$= \left(\frac{0}{0}\right) \text{ form.}$$

$$= \lim_{z\to 0}\frac{\cos z - [\cos z - z\sin z]}{2\sin z\cos z} = \lim_{z\to 0}\frac{z\sin z}{2\sin z\cos z} = \frac{0}{1} = 0$$

$$\therefore \boxed{R_1 = 0}$$

$$\therefore \oint_C \frac{dz}{z\sin z} = 2\pi i\,[\text{sum of Residue}] = 2\pi i\,[0] = 0$$

(OR)

$$f(z) = \frac{1}{z\sin z}$$

$$= \frac{1}{z\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right]} = \frac{1}{z^2\left[1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right]} \qquad \left(\because \sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right).$$

$$= \frac{1}{z^2}\left[1-\left(\frac{z^2}{6}-\frac{z^4}{120}+\cdots\right)^{-1}\right]$$

$$= \frac{1}{z^2}\left[1+\left(\frac{z^2}{6}-\frac{z^4}{120}+\cdots\right)+\left(\frac{z^2}{6}-\frac{z^4}{120}+\cdots\right)^2+\cdots\right]$$

$$= \frac{1}{z^2}\left[1+\frac{z^2}{6}-\frac{z^4}{120}+\frac{z^4}{36}+\cdots\right]$$

$$= \frac{1}{z^2}+\frac{1}{6}-\frac{z^2}{120}+\frac{z^2}{36}+\cdots$$

$$f(z) = (z-0)^{-2}+\frac{1}{6}+\frac{(z-0)^2}{120}+\frac{(z-0)^2}{36}+\cdots$$

which is the L.S Exp of $f(z)$ in powers of $(z-0)$

In L.S, the highest $-$ve power of $(z-0)$ is 2

$\therefore$ $z=0$ is a pole of order 2.

$$R_1 = \underset{\text{at } z=0}{\text{Res } f(z)} = \text{the coefficient of } \frac{1}{z} \text{ in Exp} = 0$$

$$\therefore \oint_C \frac{dz}{z\sin z} = 2\pi i\,[0] = \underline{\underline{0}}$$

(6) Evaluate $\int_C \frac{dz}{\cosh z}$: where $c: |z|=2$

Soln. Let $f(z) = \frac{1}{\cosh z} = \frac{1}{\frac{e^z+e^{-z}}{2}} = \frac{2e^z}{e^{2z}+1}$

the poles of $f(z)$ are given by $e^{2z}+1=0$

$$\Rightarrow (e^z)^2+1=0$$

$$\Rightarrow (e^z)^2-(i)^2=0$$

$$\Rightarrow (e^z+i)(e^z-i)=0$$

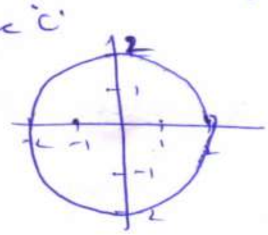
$$\Rightarrow e^z = i, -i$$

$$\Rightarrow e^z = e^{\pi i/2}, e^{-\pi i/2} \qquad \Rightarrow z = \frac{\pi i}{2}, \frac{-\pi i}{2}$$

$\therefore$ The poles of $f(z)$ are $z = \frac{\pi i}{2}, -\frac{\pi i}{2}$ which are poles of order 1

The Given curve 'c' is $|z| = 2$ which is a circle with centre $(0,0)$ & $r = 2$

Here the Two pole $z = \frac{\pi i}{2}$ & $z = -\frac{\pi i}{2}$ Lies inside curve 'c'



Residue of $f(z)$ at $z = \frac{\pi i}{2}$:

Since $z = \frac{\pi i}{2}$ is a pole of order 1

$$\therefore R_1 = \underset{\text{at } z = \frac{\pi i}{2}}{\text{Res } f(z)} = \lim_{z \to \frac{\pi i}{2}} \left(z - \frac{\pi i}{2}\right) \cdot f(z)$$

$$= \lim_{z \to i\pi/2} \left(z - \frac{\pi i}{2}\right) \cdot \frac{1}{\cosh z}$$

$$= \lim_{z \to i\pi/2} \cancel{\left(z - \frac{\pi i}{2}\right)} \cdot \frac{1}{\cancel{\left(z - \frac{\pi i}{2}\right)}\left(z + \frac{\pi i}{2}\right)}$$

$z = \frac{\pi i}{2}, -\frac{\pi i}{2}$ are the roots of $\cosh z$ $\therefore \cosh z = (\ )(\ )$

$$= \frac{1}{\frac{\pi i}{2} + \frac{\pi i}{2}} = \frac{-i}{\pi} \qquad \therefore R_1 = \frac{-i}{\pi}$$

Residue of $f(z)$ at $z = -\frac{\pi i}{2}$:

Since $z = -\frac{\pi i}{2}$ is a pole of order 1

$$\therefore R_2 = \underset{\text{at } z = -\frac{\pi i}{2}}{\text{Res } f(z)} = \lim_{z \to -\frac{\pi i}{2}} \left(z + \frac{\pi i}{2}\right) \cdot f(z)$$

$$= \lim_{z \to -\frac{\pi i}{2}} \left(z + \frac{\pi i}{2}\right) \frac{1}{\cosh z}$$

$$= \lim_{z \to -\frac{\pi i}{2}} \cancel{\left(z + \frac{\pi i}{2}\right)} \cdot \frac{1}{\left(z - \frac{\pi i}{2}\right)\cancel{\left(z + \frac{\pi i}{2}\right)}}$$

$$= \frac{i}{\pi}$$

$$\therefore R_2 = \frac{i}{\pi}$$

Hence $\oint_c f(z)\, dz = 2\pi i$ [sum of Residues]

$$\oint_c \frac{dz}{\cosh z} = 2\pi i \left(\frac{-i}{\pi} + \frac{i}{\pi} \right) = \underline{\underline{0}}$$

H/w : (1) Evaluate by Residue theorem $\int_c \frac{dz}{z^2 e^z}$ where c is $|z| = 1$ Hint: $\int_c \frac{e^{-z}}{z^2} dz$

(2) $\int \frac{\sin z}{z^6} dz$ where $c: |z| = 2$

(3) $\int_c \frac{z}{(z-1)(z-2)^2} dz$ when $c: |z-2| = \frac{1}{2}$

(4) $\int_c \frac{\sinh z}{z \cos z} dz$ when $c: |z| = \pi$ Hint $z = 0, z = (2n+1)\frac{\pi}{2}, n = 0, \pm 1, \pm 2 ---$ are poles

(5) $\int_c \frac{(2z+1)^2}{4z^3 + z} dz$ when $c: |z| = 1$

(6) $\int_c e^z/(z^2+\pi^2)^2\, dz$ when $c: |z| = 4$.

(7) $\int_c \frac{dz}{(z^2+4)^2}$ when c is $|z - i| = 2$

(8) $\int_c \frac{z \cos z}{(z - \pi/2)^3} dz$ when $c: |z-1| = 1$

* EVALUATION OF REAL DEFINITE INTEGRALS BY CONTOUR INTEGRATION

In this section, we consider the evaluation of certain types of real definite integrals. These integrals often arise in physical problems. To evaluate these integrals, we apply Residue theorem which is simpler than the usual methods of integration. The process of evaluating a definite integral by making the path of integration about a suitable contour (curve) in the complex plane is called contour integration.

TYPE I: Integrals of the type $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$ [Integration Round the unit circle]

Procedure: put $z = e^{i\theta}$ $\Rightarrow dz = e^{i\theta}\, i\, d\theta$

diff on b.s w.r.to θ

$$\Rightarrow \frac{dz}{d\theta} = ie^{i\theta} \Rightarrow \frac{dz}{ie^{i\theta}} = d\theta \Rightarrow d\theta = \frac{dz}{iz}.$$

W.K.T $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$ $\left(\because z = e^{i\theta},\ \frac{1}{z} = e^{-i\theta}\right)$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2} = \frac{z-\frac{1}{z}}{2i}.$$

also since $0 \le \theta \le 2\pi$

$\Rightarrow$ θ travels on the entire unit circle & $|z| = |e^{i\theta}| = 1$

$$\therefore \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \int_C F\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]\frac{dz}{iz}$$

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \int_C f(z)\, dz \quad (\text{say}) \quad — (1)$$

where 'C' is the unit circle $|z| = 1$.

By Residue theorem: $\int_C f(z)\,dz = 2\pi i \times [\text{sum of Residues}]$ (2)

from (1) & (2)

$$\therefore \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = 2\pi i \times [\text{sum of Residues}]$$

Problems

(1) Show by the method of residues $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$ $(a>b>0)$

(OR)

S.T $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$

Soln: we can write $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ — (1)

$0 \quad \pi \quad 2\pi$

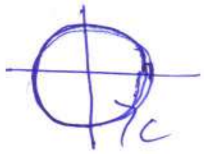
$\frac{1}{2}$ of $[0,2\pi]$ $= [0,\pi]$

Let 'C' be the unit circle: ie $C: |z|=1$

put $z = e^{i\theta}$ diff w.r.t. θ $\Rightarrow \frac{dz}{d\theta} = ie^{i\theta} = iz$

$\Rightarrow d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$



~~$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i} =$~~ sub all above values in Eqn(1) then

$\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_C \frac{1}{a+b\left[\frac{z^2+1}{2z}\right]}\frac{dz}{iz}$

$= \frac{\cancel{1}}{\cancel{2}}\frac{1}{i}\int_C \frac{\cancel{2}}{bz^2+2az+b}dz$

$\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{i}\int_C \frac{dz}{bz^2+2az+b} =$ ~~$\frac{2}{i}\oint f(z)\,dz$~~ — (2)

~~where~~ let $f(z) = \frac{1}{bz^2+2az+b}$

The poles of $f(z)$ are given by $bz^2+2az+b=0$

$z = \frac{-2a \pm \sqrt{4a^2-4\cdot b\cdot b}}{2\cdot b}$

$z = \frac{-2a \pm 2\sqrt{a^2-b^2}}{2b}$

$z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$

$bz^2+2az+b=0$
$\downarrow a \quad \downarrow b \quad \downarrow c$
$ax^2+bx+c=0$

$\therefore$ The poles of $f(z)$ are $z = \frac{-a+\sqrt{a^2-b^2}}{b}, \frac{-a-\sqrt{a^2-b^2}}{b}$

which are poles of order '1'

Let $\alpha = \dfrac{-a+\sqrt{a^2-b^2}}{b}$ & $\beta = \dfrac{-a-\sqrt{a^2-b^2}}{b}$

since $a>b>0$

$\Rightarrow |\beta| > 1 \Rightarrow 1 > \dfrac{1}{|\beta|} \Rightarrow \dfrac{1}{|\beta|} < 1$

But ~~the~~ w.k.t product of the roots = $\dfrac{c}{a} = \dfrac{b}{b} = 1$

i.e $\alpha \cdot \beta = 1$

| $bz^2+2az+b=0$

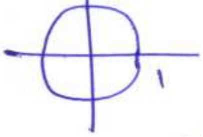
$\Rightarrow |\alpha\beta| = 1$

$\Rightarrow |\alpha||\beta| = 1$

$\Rightarrow |\alpha| = \dfrac{1}{|\beta|} < 1$

$\Rightarrow |\alpha| < 1$

$\therefore$ 'α' lies inside the unit circle 'c'



i.e the pole $\alpha = \dfrac{-a+\sqrt{a^2-b^2}}{b}$ lies inside 'c' & the pole $\beta = \dfrac{-a-\sqrt{a^2-b^2}}{b}$ lies outside 'c'

Residue of $f(z)$ at $z=\alpha$. ~~$z = \frac{-a+\sqrt{a^2-b^2}}{b}$~~ : since $z=\alpha$ ~~$z = \frac{-a+\sqrt{a^2-b^2}}{b}$~~ is a pole of order '1'

$\therefore$ ~~Res~~ $R_1 = \underset{\text{at } z=\alpha}{\text{Res } f(z)} = \lim\limits_{z\to\alpha} (z-\alpha)\cdot f(z)$

$= \lim\limits_{z\to\alpha} (z-\alpha) \cdot \dfrac{1}{bz^2+2az+b}$

| If $z=a$ is a pole of order '1' then $\underset{\text{at } z=a}{\text{Res } f(z)} = \lim\limits_{z\to a} (z-a) f(z)$

$= \lim\limits_{z\to\alpha} \cancel{(z-\alpha)} \cdot \dfrac{1}{b\cancel{(z-\alpha)}(z-\beta)}$

$= \dfrac{1}{b(\alpha-\beta)}$

$= \dfrac{1}{b\left[\dfrac{-a+\sqrt{a^2-b^2}}{b} + \dfrac{a+\sqrt{a^2-b^2}}{b}\right]}$

$R_1 = \dfrac{1}{2\sqrt{a^2-b^2}}$

| If α, β are the roots of $ax^2+bx+c=0$ then $ax^2+bx+c = a(x-\alpha)(x-\beta)$
Here α, β are the roots of $bz^2+2az+b=0$
$\therefore bz^2+2az+b = b(z-\alpha)(z-\beta)$

By Residue theorem.

$\therefore \displaystyle\int_c f(z)\,dz = \int_c \dfrac{1}{bz^2+2az+b}\,dz = 2\pi i \times [\text{sum of Residues}]$

$= \cancel{2}\pi i \times \dfrac{1}{\cancel{2}\sqrt{a^2-b^2}}$ —— (3)

Sub ③ in ②

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\cancel{1}}{\cancel{i}} \times \cancel{2}\pi \cancel{i} \times \frac{1}{\cancel{2}\sqrt{a^2-b^2}} = \frac{\cancel{2}\pi}{\sqrt{a^2-b^2}} \quad \text{— ④}$$

~~now sub ④ in ①~~

$$\therefore \Rightarrow \int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{\cancel{2}} \times \frac{\cancel{2}\pi}{\sqrt{a^2-b^2}} = \frac{\pi}{\sqrt{a^2-b^2}} \;\checkmark //$$

② P.T $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\, d\theta = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\,\right]$ where $a>b>0$

<u>Soln</u>: It is required to Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\, d\theta$

we can write $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\, d\theta = \int_0^{2\pi} \frac{(1-\cos 2\theta)/2}{a+b\cos\theta}$ $\quad \left(\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$

$$= \int_0^{2\pi} \frac{1-\cos 2\theta}{2a+2b\cos\theta}\, d\theta$$

$\cos 2\theta = R.P(e^{i2\theta})$

$$\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\, d\theta = R.P \text{ of } \int_0^{2\pi} \frac{1-e^{2i\theta}}{2a+2b\cos\theta}\, d\theta \quad \text{— (1)}$$

put $z = e^{i\theta}$

diff on B.S w.r.to 'θ'

$$\frac{dz}{d\theta} = ie^{i\theta} \Rightarrow \frac{dz}{ie^{i\theta}} = d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

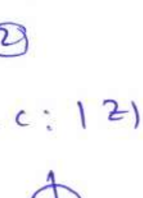
$$\& \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z} \quad \left(\because e^{-i\theta} = \frac{1}{e^{i\theta}} = \frac{1}{z}\right)$$

Sub all above values in (1)

$$\Rightarrow \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\, d\theta = R.P \text{ of } \int_0^{2\pi} \frac{1-z^2}{\left(2a+\cancel{2}b\frac{z^2+1}{\cancel{2}z}\right)} \frac{dz}{iz} \quad \left(\because e^{2i\theta} = (e^{i\theta})^2 = z^2\right)$$

$$\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\, d\theta = R.P \frac{1}{i}\int_C \frac{1-z^2}{bz^2+2az+b}\, dz \quad \text{— ②}$$

where $C: |z| = 1$



Now consider $\int_C \frac{1-z^2}{bz^2+2az+b}\,dz$

Here $f(z) = \dfrac{1-z^2}{bz^2+2az+b}$ & curve 'C' is $|z|=1$

The poles of $f(z)$ are given by $(b)z^2+(2a)z+b=0$

$$z = \frac{-2a \pm \sqrt{(2a)^2-4\cdot b\cdot b}}{2\cdot b}$$

$ax^2+bx+c=0$

$x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$z = \dfrac{-a+\sqrt{a^2-b^2}}{b}$, $\dfrac{-a-\sqrt{a^2-b^2}}{b}$ are the poles of order '1'

let $\alpha = \dfrac{-a+\sqrt{a^2-b^2}}{b}$, $\beta = \dfrac{-a-\sqrt{a^2-b^2}}{b}$

since $a>b>0$, $|\beta|>1$ & $\alpha\beta=1 \Rightarrow |\alpha|<1 \Rightarrow$ 'α' lies inside curve $c:|z|=1$

$\therefore$ out of these two poles 'α' only lies inside the curve 'C': $|z|=1$

<u>Residue of $f(z)$ at $z=\alpha$</u>:

since 'α' is a pole of order '1'

$$\therefore R_1 = \left[\operatorname{Res} f(z)\right]_{\text{at } z=\alpha} = \lim_{z\to\alpha} (z-\alpha)\cdot f(z)$$

$$= \lim_{z\to\alpha} (z-\alpha)\cdot \frac{(1-z^2)}{bz^2+2az+b}$$

$$= \lim_{z\to\alpha} \cancel{(z-\alpha)}\cdot \frac{(1-z^2)}{b\cancel{(z-\alpha)}(z-\beta)}$$

$\because \alpha$ & β are the roots of $bz^2+2az+b=0$

$\Rightarrow bz^2+2az+b = (z-\alpha)(z-\beta)$

$$= \frac{1-\alpha^2}{b(\alpha-\beta)}$$

$$= \frac{\alpha\left(\frac{1}{\alpha}-\alpha\right)}{b(\alpha-\beta)}$$

$$= \frac{\alpha(\beta-\alpha)}{b(\alpha-\beta)} \qquad (\because \alpha\beta=1)$$

$$R_1 = \frac{-\alpha}{b} = -\left[\frac{-a+\sqrt{a^2-b^2}}{b^2}\right]$$

$\therefore$ By Cauchy Residue theorem.

$$\int_C f(z)\,dz = 2\pi i \times [\text{Sum of Residues}]$$

$$\int_C \frac{1-z^2}{bz^2+2az+b}\,dz = -2\pi i \times \left[\frac{-a+\sqrt{a^2-b^2}}{b^2}\right] \quad \text{—— (3)}$$

sub (3) in (2)

$$\Rightarrow \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = R.P\ \frac{1}{\cancel{i}}\left[-2\pi \cancel{i}\left(\frac{-a+\sqrt{a^2-b^2}}{b^2}\right)\right]$$

$$= \frac{2\pi}{b^2}\left[+a-\sqrt{a^2-b^2}\right]$$

Note: Taking $a=6$ & $b=3$ in above problem. we get

$$\int_0^{2\pi} \frac{\sin^2\theta}{6+3\cos\theta}\,d\theta = \frac{2\pi}{9}(2-\sqrt{3})$$

H/w: (1) S.T $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$, $a>b>0$ using Residue theorem.

(2) Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$ using Residue theorem

(3) S.T $\int_0^{2\pi} \frac{1+4\cos\theta}{17+8\cos\theta}\,d\theta = 0$

(4) Evaluate $\int_0^{2\pi} \frac{\sin 3\theta}{5-3\cos\theta}\,d\theta$ using Residue theorem.

Hint: put $z=e^{i\theta}$: $\sin 3\theta = \frac{e^{i3\theta}-e^{-i3\theta}}{2i} = \frac{(e^{i\theta})^3-(e^{i\theta})^{-3}}{2i} = \frac{z^3-z^{-3}}{2i} = \frac{z^3+\frac{1}{z^3}}{2i}$

(5) Evaluate $\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2}$ using Residue theorem.

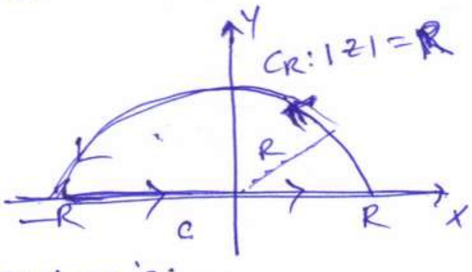
(6)

TYPE II.

Integrals of the type $\int_{-\infty}^{\infty} f(x)\,dx$: [Integration around semicircle]

To solve the integrals of the type $\int_{-\infty}^{\infty} f(x)\,dx$, we consider $\int_C f(z)\,dz$; where 'c' is the closed contour consisting of the semi-circle $C_R : |z| = R$ in upper half plane, together with the real axis from $-R$ to R.

[ie $C = C_R \cup$ real axis from $-R$ to R]



If $f(z)$ has no ~~singular point~~ poles on the real axis & ~~some~~ but if $f(z)$ has some poles inside contour 'C' then By Residue theorem.

$$\int_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i\,[\text{sum of Residues at Interior poles}]$$

when Radius$(R) \to \infty$, then the semicircle C_R will become a Big semicircle, then the all poles lies inside 'c'.

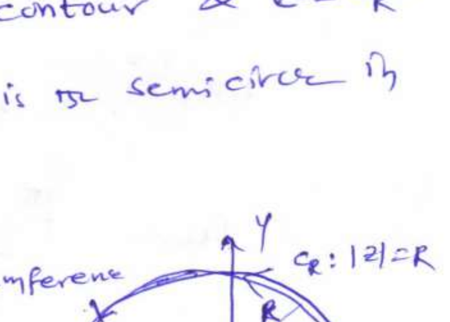
$$= 2\pi i \times \{\text{Sum of Residues at Interior poles}\}$$

Here we prove that $\int_{C_R} f(z)\,dz = 0$ as $|z| \to \infty$

TYPE II: Integrals of the type $\int_{-\infty}^{\infty} f(x)\,dx$ [Integration around semi circle]

To solve the integrals of the type $\int_{-\infty}^{\infty} f(x)\,dx$, we consider $\int_{-\infty}^{\infty} f(x)\,dx = \int_C f(z)\,dz$ where 'c' is the closed contour & $C = C_R \cup$

$C = C_R \cup$ Real axis from $-R$ to R [C_R is the semicircle in the upper half plane with radius R]



If $f(z)$ has no poles on real axis & on circumference ~~for on a circle~~. but $f(z)$ has some poles inside curve 'c'. Then By Residue theorem

$$\int_C f(z)\,dz = 2\pi i \times [\text{Sum of the Residues at Interior Poles}]$$

$$\Rightarrow \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \times [\text{Sum of Residues at Interior poles}]$$

$(\because C = C_R \cup [\text{real axis from } -R \text{ to } R]$

Here we S.T $\int_{C_R} |f(z)|\,dz \to 0$ as $R \to \infty$

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \times$ [sum of Residues at interior poles]

<u>Note:</u> Radius R is taken so large that all the singularities of $f(z)$ (Poles) lie within semicircle C_R.

<u>Prob (1)</u>. Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$

<u>Soln</u>.

Here $f(x) = \frac{1}{(x^2+a^2)^2}$

$f(-x) = \frac{1}{((-x)^2+a^2)^2} = \frac{1}{(x^2+a^2)^2} = f(x)$

$\therefore f(x)$ is an Even function

$$\int_{-\infty}^{\infty} f(x)\,dx = \begin{cases} 2\int_0^{\infty} f(x)\,dx & \text{if } f \text{ is Even} \\ 0 & \text{if } f \text{ is odd function} \end{cases}$$

$\Rightarrow \int_0^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$

$\Rightarrow \int_0^{\infty} \frac{1}{(x^2+a^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\,dx \quad -(1)$

Now let $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\,dx = \int_C f(z)\,dz$ where $f(z) = \frac{1}{(z^2+a^2)^2}$

& 'C' is the contour consisting of the semi-circle C_R of radius R together with the real axis from $-R$ to R

The poles of $f(z)$ are given by $(z^2+a^2)^2 = 0$

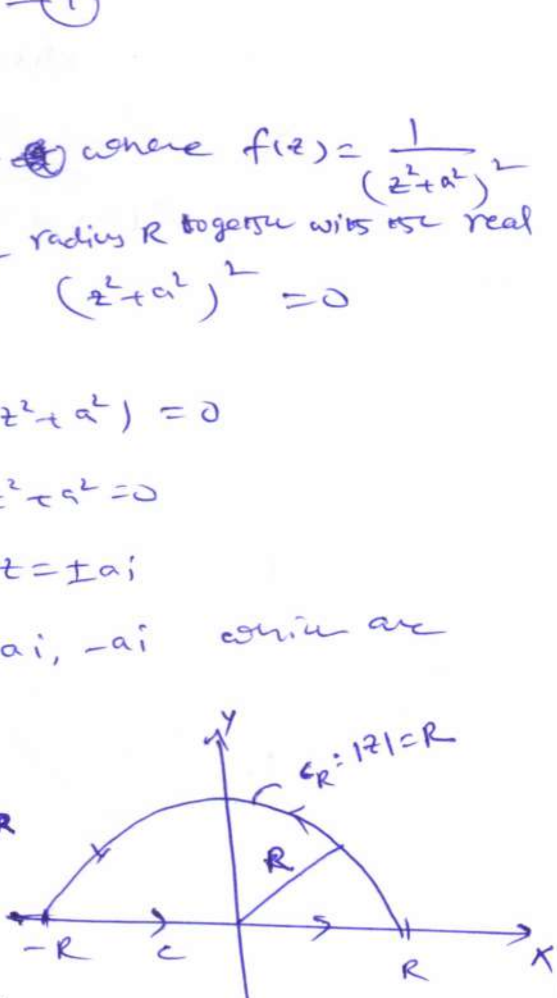
$\Rightarrow (z^2+a^2)(z^2+a^2) = 0$

$\Rightarrow z^2+a^2 = 0$ (or) $z^2+a^2 = 0$

$\Rightarrow z = \pm ai$ (or) $z = \pm ai$

$\therefore$ The poles are $z = ai, ai, -ai, -ai$ which are poles of order 2

The only pole $z = ai$ lies inside semi circle C_R



$C: C_R \cup$ [x-axis from $-R$ to R]

<u>Residue of $f(z)$ at $z = ai$</u>.

Since $z = ai$ is a pole of order 2

$R_1 = \left(\text{Res } f(z)\right]_{\text{at } z=ai} = \frac{1}{1!} \lim_{z \to ai} \frac{d}{dz}(z-ai)^2 f(z)$

$$= \lim_{z \to ai} (z-ai)^2 \cdot \frac{1}{(z^2+a^2)^2}$$

$$= \lim_{z \to ai} \frac{d}{dz}\left[\cancel{(z-ai)^2} \cdot \frac{1}{\cancel{(z-ai)^2}(z+ai)^2}\right] \qquad (\because \text{ since } z = ai, -ai \text{ } \sout{\text{multiple}} \text{ roots of } (z^2+a^2)^2$$

$$= \lim_{z \to ai} \frac{-2}{(z+ai)^3}$$

$$= \frac{-2}{(ai+ai)^3}$$

$$= \frac{-2}{(2ai)^3}$$

$$R_1 = \frac{1}{4a^3 i}$$

Hence by Residue theorem.

$\therefore \int_C f(z)\, dz = 2\pi i \times [\text{sum of the residues at interior poles}]$

$$= \cancel{2}\pi \cancel{i} \times \left[\frac{1}{\cancel{4}a^3 \cancel{i}}\right] = \frac{\pi}{2a^3}$$

$$\Rightarrow \quad \int_C f(z)\, dz = \frac{\pi}{2a^3}$$

$$\Rightarrow \quad \int_{C_R} f(z)\, dz + \int_{-R}^{R} f(x)\, dx = \frac{\pi}{2a^3} \quad \text{——} (2) \qquad (\because C = C_R \cup \text{Real Axis from } -R \text{ to } R)$$

Now we S.T $\int_{C_R} f(z)\, dz \to 0$ as $R \to \infty$

take

Now $$\left| \int_{C_R} \frac{dz}{(z^2+a^2)^2} \right| \le \int_{C_R} \frac{|dz|}{|(z^2+a^2)^2|}$$

$$\le \frac{1}{(R^2-a^2)^2} \int_0^{\pi} R\, d\theta$$

$$= \frac{R}{(R^2-a^2)^2} (\theta)_0^{\pi}$$

$$= \frac{R\pi}{(R^2-a^2)^2} = \frac{\cancel{R}\pi}{(R^2)^{\cancel{2}}\left[1-\frac{a^2}{R^2}\right]^2}$$

$(\because$ & $z = Re^{i\theta}$

$\Rightarrow dz = iRe^{i\theta} d\theta$

$\Rightarrow |dz| = R\, d\theta$

$|z|^2 = |Re^{i\theta}|^2 = R^2$

$|z^2+a^2| = |z^2-(-a^2)| \ge |z|^2 - |-a|^2 = |z|^2 - a^2 = R^2-a^2$

$|z^2+a^2| \ge R^2-a^2$

$$\frac{1}{|z^2+a^2|} < \frac{1}{R^2-a^2}$$

as $R \to \infty$

$$\left| \int_{C_R} \frac{dz}{(z^2+a^2)^2} \right| \to 0$$

$$\Rightarrow \int_{C_R} \frac{dz}{(z^2+a^2)^2} \to 0 \text{ as } R \to \infty \quad \Rightarrow \int_{C_R} f(z)\,dz \to 0 \text{ as } R \to \infty \quad (3)$$

Sub (3) in (2)

from ~~z(2)~~ ~~now~~ $\because$ Eqn (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{2a^3} \quad (\because R \to \infty)$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\,dx = \frac{\pi}{2a^3} \quad (4)$$

Sub (4) in (1)

$$\Rightarrow \int_0^{\infty} \frac{1}{(x^2+a^2)^2} = \frac{1}{2} \times \frac{\pi}{2a^3} = \frac{\pi}{4a^3}$$

Note. Evaluate $\int_0^{\infty} \frac{dx}{(1+x^2)^2}$ using Residue theorem.

put $a=1$ in the above problem. then we get

~~then~~ $\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$

Prob (2). Using the method of contour Integration,

prove that $\int_0^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$ (OR) Evaluate $\int_0^{\infty} \frac{1}{x^6+1}\,dx$

Using Residue theorem.

Soln. ~~Since integrand is an even function~~

i.e. let $f(x) = \frac{1}{x^6+1}$

$$\Rightarrow f(-x) = \frac{1}{(-x)^6+1} = \frac{1}{x^6+1} = f(x)$$

$\therefore f(x)$ is an Even function

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx = 2\cdot\int_{0}^{\infty} \frac{1}{x^6+1}\,dx$$

$\left|\ \int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx \right.$ if $f(x)$ is even

$$\Rightarrow \int_0^{\infty} \frac{1}{x^6+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x^6+1}\,dx \qquad (1)$$

consider $\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx = \int_C f(z)\,dz$ where 'C' is the contour consisting of the semi-circle C_R of radius R together with the real axis from $-R$ to R. and $f(z) = \frac{1}{z^6+1}$

The poles of $f(z)$ are given by $z^6+1=0$

$\Rightarrow z^6 = -1$ | No. of poles $= 6$

$\Rightarrow z = (-1)^{1/6}$

$\Rightarrow z = [\cos\pi + i\sin\pi]^{1/6}$

$\Rightarrow z = [\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)]^{1/6}$

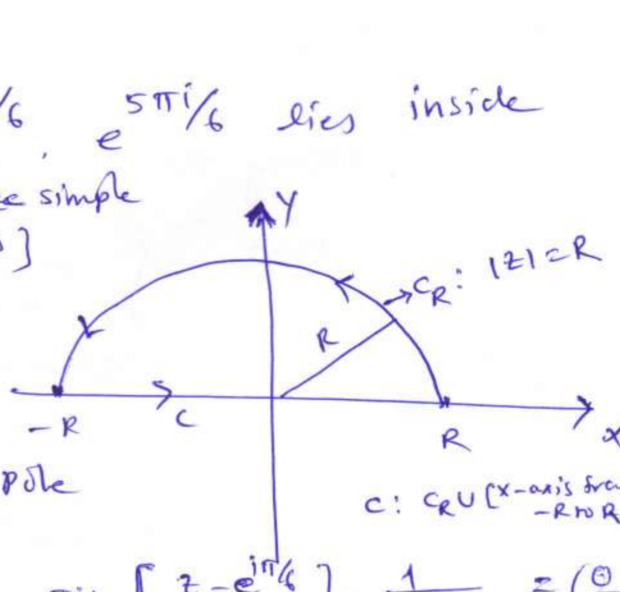
$z = \cos\frac{(2n+1)\pi}{6} + i\sin\frac{(2n+1)\pi}{6}$ | $(\cos\theta + i\sin\theta)^{n} = \cos n\theta + i\sin n\theta$

when $n = 0,1,2,3,4,5$

$\Rightarrow z = e^{\frac{(2n+1)i\pi}{6}}$ where $n = 0,1,2,3,4,5$

$\Rightarrow z = e^{i\pi/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$ which are the poles of $f(z)$.

The only poles $z = e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}$ lies inside the semi-circle, C_R. [which are simple poles]



c: $C_R \cup$ [x-axis from $-R$ to R]

Residue of $f(z)$ at $z = e^{\pi i/6}$

since $z = e^{\pi i/6}$ is a simple pole

$$\therefore R_1 = \operatorname{Res}_{z=e^{\pi i/6}}[f(z)] = \lim_{z\to e^{\pi i/6}} \left[z - e^{i\pi/6}\right]\cdot\frac{1}{z^6+1} = \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to e^{\pi i/6}} \frac{1}{6z^5} \quad \text{(By L'Hospital Rule)}$$

$$R_1 = \frac{1}{6}e^{-5\pi i/6}$$

Residue of $f(z)$ at $z = e^{3\pi i/6}$:

since $z = e^{3\pi i/6}$ is a simple pole

$$\therefore R_2 = \underset{at\ z=e^{\frac{3\pi i}{6}}}{Res\,[f(z)]} = \lim_{z\to e^{\frac{3\pi i}{6}}} \left(z - e^{\frac{3\pi i}{6}}\right) \cdot \frac{1}{z^6+1} = \frac{0}{0} \text{ (form)}$$

$$= \lim_{z\to e^{\frac{3\pi i}{6}}} \frac{1}{6z^5}$$

$$R_2 = \frac{1}{6} e^{-5\pi i/2}$$

Residue of $f(z)$ at $z = e^{5\pi i/6}$:

since $z = e^{\frac{5\pi i}{6}}$ is a simple pole.

$$\therefore R_3 = \underset{at\ z=e^{\frac{5\pi i}{6}}}{Res\,[f(z)]} = \lim_{z\to e^{5\pi i/6}} \left[z - e^{\frac{5\pi i}{6}}\right] \cdot \frac{1}{z^6+1} = \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to e^{\frac{5\pi i}{6}}} \frac{1}{6z^5}$$

$$R_3 = \frac{1}{6} e^{-25\pi i/6}$$

Hence By Residue theorem.

$$\therefore \int_C f(z)\,dz = 2\pi i\,[\text{ sum of the Residues at poles}]$$

$$= 2\pi i\left[\frac{1}{6} e^{-5\pi i/6} + \frac{1}{6} e^{-5\pi i/2} + \frac{1}{6} e^{-25\pi i/6}\right]$$

$$= \frac{2\pi i}{6}\left[e^{-5\pi i/6} + e^{-5\pi i/2} + e^{-25\pi i/6}\right]$$

$$= \frac{\pi i}{3}\left[\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right) + \left(\cos\frac{5\pi}{2} - i\sin\frac{5\pi}{2}\right) + \left(\cos\frac{25\pi}{6} - i\sin\frac{25\pi}{6}\right)\right]$$

$$\int_C f(z)\,dz = \frac{2\pi}{3}$$

Since $c = C_R \cup (\text{Real axis from } -R \text{ to } R)$

$$\Rightarrow \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = \frac{2\pi}{3} \quad \text{--- (2)}$$

But $\int_{C_R} f(z)\,dz = \int_{C_R} \frac{1}{z^6+1}\,dz \to 0$ as $R \to \infty$ } — (3)

Sub (3) in (2) then

Eqn (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{2\pi}{3}$.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx = \frac{2\pi}{3} \quad \text{— (4)}$$

Now sub (4) in (1)

$$\Rightarrow \int_{0}^{\infty} \frac{1}{x^6+1}\,dx = \frac{1}{\cancel{2}} \times \frac{\cancel{2}\pi}{3} = \frac{\pi}{3}$$

$$\Rightarrow \int_{0}^{\infty} \frac{1}{x^6+1}\,dx = \frac{\pi}{3}$$

Pro (3). Evaluate $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$ using Residue theorem.

Soln: Consider $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \int_C f(z)\,dz$ — (1)

where $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$ and 'C' is the contour consisting of the semi circle C_R of radius R together with the real axis from $-R$ to R.

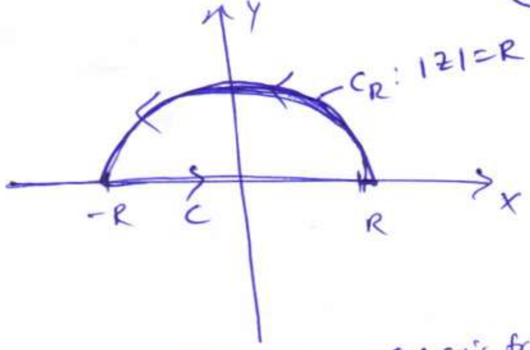
The poles of $f(z)$ are given by. $(z^2+1)(z^2+4) = 0$

$$\Rightarrow z^2+1 = 0 \text{ (or) } z^2+4 = 0$$

$$\Rightarrow z = \pm i \text{ (or) } z = \pm 2i$$

$\therefore z = i, -i, 2i, -2i$ which are the poles of $f(z)$.

The only poles $z = i, 2i$ which lies inside the semi circle C_R.



curve C : $C_R \cup$ [x axis from $-R$ to R]

Residue of $f(z)$ at $z = i$

Since $z = i$ is a simple pole.

$$\therefore R_1 = \underset{\text{at } z=i}{\text{Res}\,[f(z)]} = \underset{z \to i}{\text{Lt}} \, [z - i] \cdot \frac{z^2}{(z^2+1)(z^2+4)}$$

$$= \underset{z \to i}{\text{lt}} \, (\cancel{z - i}) \cdot \frac{z^2}{(\cancel{z-i})(z+i)(z+2i)(z-2i)}$$

$$R_1 = -\frac{1}{6i}$$

Residue of $f(z)$ at $z = 2i$:

Since $z = 2i$ is a simple pole.

$$\therefore R_2 = \underset{\text{at } z=2i}{\text{Res}\,[f(z)]} = \underset{z \to 2i}{\text{lt}} \, (z - 2i) \cdot \frac{z^2}{(z^2+1)(z^2+4)}$$

$$= \underset{z \to 2i}{\text{lt}} \, (\cancel{z - 2i}) \cdot \frac{z^2}{(z-i)(z+i)(\cancel{z-2i})(z+2i)}$$

$$R_2 = \frac{1}{3i}$$

Hence By Residue theorem

$$\int_C f(z)\, dz = 2\pi i \times [\text{Sum of the Residues}]$$

$$= 2\pi i \times \left[-\frac{1}{6i} + \frac{1}{3i} \right] = \frac{\pi}{3}$$

$$\therefore \int_C f(z)\, dz = \frac{\pi}{3}$$

Since curve $C = C_R \cup$ [Real axis from $-R$ to R]

$$\Rightarrow \int_{C_R} f(z)\, dz + \int_{-R}^{R} f(x)\, dx = \frac{\pi}{3} \quad \text{— (2)}$$

But $\int_{C_R} f(z)\,dz = \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\,dz \to 0$ as $R \to \infty$ } → (3)

Sub (3) in (2) then

Eqn (2) $\Rightarrow$ $\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{3}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{3} \quad (4)$$

H/w :

(1) Evaluate By contour Integration $\int_0^{\infty} \frac{1}{1+x^2}\,dx$

Ans : $\frac{\pi}{2}$

(2) Evaluate $\int_0^{\infty} \frac{\log x}{1+x^2}\,dx$

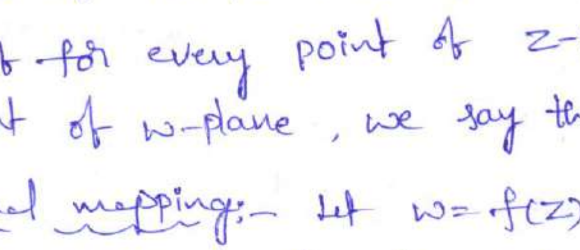
(3) P.T $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx = \frac{5\pi}{12}$

Transformation:- Let $z = x+iy$ in z-plane and the corresponding $w = u+iv$ in w-plane. We can establish a graphical correspondence between the two planes is called Transformation (or) mapping.

The corresponding points w in the w-plane are called the "Images" of the points z of the z-plane.

If for every point of z-plane, there corresponds one and only one point of w-plane, we say that the correspondence is one-to-one

Conformal mapping:- Let $w = f(z)$ be a transformation which maps the curves C_1, C_2 intersecting at 'P' of the z-plane into the curves C_1 & C_2 intersecting at 'Q' of the w-plane respectively



Then if the angle b/w C_1 & C_2 at P is equal both in magnitude and sense to angle b/w C_1 & C_2 at Q. i.e the mapping is said to be conformal.

Elementary Transformation:-

Translation:- Let $w = z+c$

The transformation $w = z+c$ translates 'z' through the constant 'c' for all $z \in C$

Let $w = u+iv$ and $c = c_1+ic_2$ where c_1, c_2 are constants

Then $w = z+c = (x+c_1)+i(y+c_2)$

$\therefore$ The point (x,y) of the z-plane is mapped to the point (u,v) (or) $(x+c_1, x+c_2)$ of the w-plane.

Rotation:- Let $w = cz$, where c is the complex constant

Let $z = re^{i\theta}$, $w = Re^{i\phi}$ and $c = \beta e^{i\alpha}$

Then $w = cz \Rightarrow Re^{i\phi} = \beta r e^{i(\theta+\alpha)}$

we get $R = \beta r$ & $\phi = \theta + \alpha$

Thus under this transformation a point $P(r, \theta)$ in the z-plane is mapped to the point $P'(\beta r, \theta+\alpha)$ in the w-plane. Thus, this transformation effects an expansion when $|c|>1$ and a contraction when $0<|c|<1$ of the radius vector by $\beta = |c|$ and rotation through an angle $\alpha = \arg c$.

Hence any figure in z-plane is transformed into, geometrically a similar figure in the w-plane.

In particular circles are mapped to circles.

Magnification:- Let $w = cz$ be a transformation $\rightarrow (1)$

$$\text{Let } w = Re^{i\phi},\ z = re^{i\theta}$$

$$(1) \Rightarrow Re^{i\phi} = cre^{i\theta}$$

$$R = cr,\ \phi = \theta$$

i.e The point (r, θ) is mapped into the point (r, θ)

i.e $w = cz$ represent a magnification (or) contraction according as $b>1$ (or) $b<1$

Inversion:-

Critical point:- The points where $\frac{dw}{dz}=0$ (or) ∞ are called critical points and the points $\frac{dw}{dz}\neq 0$ are called ordinary points 2

Ex:- (1) consider $f(z)=z^2$, It has a critical point at $z=0$

Theorem:- Let $f(z)$ be an analytic function of z in a domain D of the z-plane and let $f'(z)\neq 0$ in D. Then $w=f(z)$ is a conformal mapping at all points of D

Proof:- Let $P(z)$ be a point in the region R of the z-plane and $P'(w)$ is the corresponding point in the region R' of the w-plane

Suppose z moves on a curve C and w moves on the corresponding curve C'.

Let $Q(z+\Delta z)$ be a neighbouring point on C and $Q'(w+\Delta w)$ be the corresponding point on C'.

Suppose $\overline{PQ}=\Delta z$

$P'Q'=\Delta w$. Then $\Delta z = re^{i\theta}$ where $|\Delta z| = r$

θ = amplitude with x-axis

Similarly $\Delta w = r'e^{i\theta'}$ where $|\Delta w| = r'$

θ' = amplitude with u-axis

we can have $\frac{\Delta w}{\Delta z} = \frac{r'}{r}e^{i(\theta-\theta')}$

Suppose the tangents to the curves C and C'

At P and P' makes angles α and α' with x-axis & U-axis resp

As $\Delta z\to 0$, $\theta\to\alpha$

and $\theta'\to\alpha'$

$$\therefore f'(z) = \frac{dw}{dz} = \lim_{\substack{\theta\to\alpha\\ \theta'\to\alpha'}}\left(\frac{r'}{r}e^{i(\theta'-\theta)}\right) = \lim\left(\frac{r'}{r}\right)e^{i(\alpha'-\alpha)} \longrightarrow (1)$$

Let β and ϕ be the modulus and amplitude of the function $f'(z)\neq 0$. Then $f'(z) = \beta e^{i\phi} \longrightarrow (2)$

From ① & ②, $\beta = \psi + \frac{\pi'}{\pi} \longrightarrow ③$

$\phi = \alpha' - \alpha \longrightarrow ④$

Let C_1 be another curve through p in the z-plane and C_1' be the corresponding curve through p' in the w-plane.
Let the tangents at p and p' to the curves C_1 and C_1' make angles β and β' with x-axis and u-axis.

Then, we have $\phi = \beta' - \beta \longrightarrow ⑤$

from ④ + ⑤, we get

$\alpha' - \alpha = \beta' - \beta$

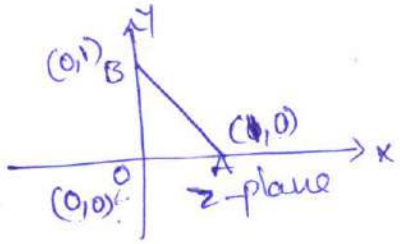
$\Rightarrow \beta - \alpha = \beta' - \alpha' = V$

Thus, we can see the angle between curves and their images C' and C_1' is preserved in magnitude and direction through the C and C_1 transformation.

Hence proved.

① Find and plot the image of the triangular region with vertices of $(0,0)$, $(1,0)$, $(0,1)$ under the transformation $w=(1-i)z+3$?

<u>Sol:-</u> Given transformation is

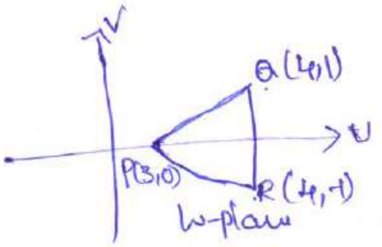


$$w=(1-i)z+3$$

$$=(1-i)(x+iy)+3$$

$$w=(x+y+3)+i(y-x) \longrightarrow ①$$

$$w=\ u\ +iv$$



where $u=x+y+3$.

$v=y-x$

If At the point $(x,y)=(0,0)$ is mapped $u=3, v=0 \Rightarrow (u,v)=(3,0)$

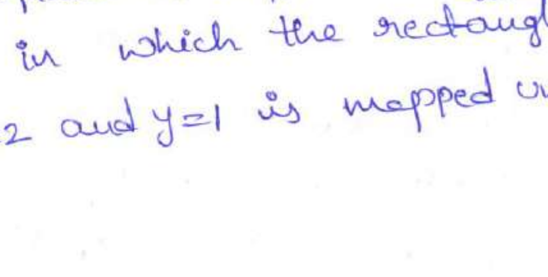
At the point $(x,y)=(1,0)$ " " $u=4, v=-1 \Rightarrow (u,v)=(4,-1)$

At the point $(x,y)=(0,1)$ " " $u=4, v=1 \Rightarrow (u,v)=(4,1)$

Hence the given point triangle in the z-plane is mapped into w-plane is triangle.

② Find the region in the w-plane in which the rectangle bounded by the lines $x=0, y=0, x=2$ and $y=1$ is mapped under the transformation $w=z+(2+3i)$

<u>Sol:-</u> Given transformation is



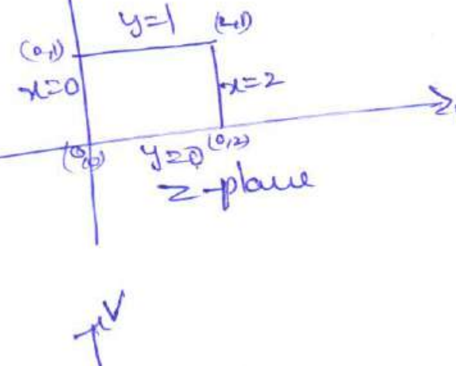
$$w=z+(2+3i)$$

$$=(x+iy)+(2+3i)$$

$$w=(x+2)+i(y+3)$$

$$w=\ u\ +iv$$

Here $\left.\begin{array}{l} u=x+2 \\ v=y+3 \end{array}\right\} \longrightarrow ①$



Now

At $x=0 \Rightarrow u=2$

At $y=0 \Rightarrow v=3$

At $x=2 \Rightarrow u=4$

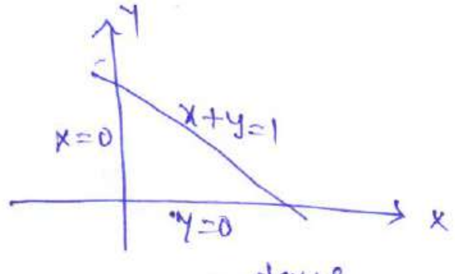
At $y=1 \Rightarrow v=4$

Hence rectangle in the z-plane is mapped into the rectangle in w-plane.

(3) Find the image of the triangular region in the z-plane bounded by the lines $x=0, y=0$ & $x+y=1$ under the transformation $w=2z$

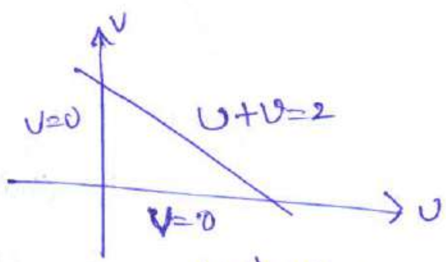
Sol:- Given transformation is

$$w = 2z$$
$$w = 2x + i2y$$
$$w = u + iv$$



Here $u = 2x$
$v = 2y$

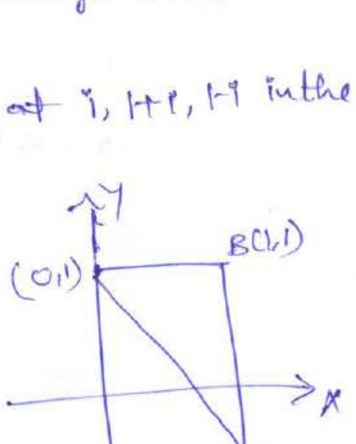
Now
At $x=0 \Rightarrow u=0$
At $y=0 \Rightarrow v=0$
At $x+y=1 \Rightarrow \frac{u}{2}+\frac{v}{2}=1$
$\Rightarrow u+v=2$



Thus we have shown that the given mapping $w=2z$ transforms the triangle in z-plane onto the triangle in the w-plane bounded by $u=0, v=0$ and $u+v=2$

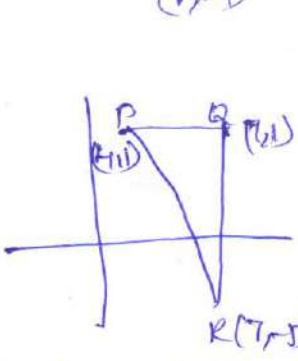
(4) Find the image of the triangle with vertices at $i, 1+i, 1-i$ in the z-plane under the transformation $w = 3z+4-2i$

Sol:- The given transformation $w = 3z+4-2i$



$$w = 3(x+iy)+4-2i$$
$$w = (3x+4)+i(3y-2)$$
$$= u+iv$$

Here $\left.\begin{array}{l} u = 3x+4 \\ v = 3y-2 \end{array}\right\} \rightarrow (1)$



Now At $i = 0+1\cdot i$
$\Rightarrow (x,y) = (0,1) \Rightarrow u=4, v=1$ i.e $(u,v)=(4,1)$

At $1+i = x+iy$
$\Rightarrow (x,y)=(1,1) \Rightarrow u=7, v=1$ i.e $(u,v)=(7,1)$

At $1-i = x+iy$
$\Rightarrow (x,y)=(1,-1) \Rightarrow u=7, v=-5$ i.e $(u,v)=(7,-5)$

$\therefore$ Hence the triangle ABC in z-plane is transformed to the triangle PQR in w-plane under the given transformation.

(5) Under the transformation $w = \frac{1}{z}$ find the image of the circle $|z-2i| = 2$ 4

Sol: The given transformation can be written as

$$z = \frac{1}{w}$$

$$x+iy = \frac{1}{u+iv} = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$$

$$= \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

$$\therefore x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

$|z-2i| = 2$ is a circle in z-plane passing through origin

$$\therefore |x+iy-2i|^2 = x^2+(y-2)^2 = 2^2$$

$|x+iy| = a$

$\sqrt{x^2+y^2} = a$

$x^2+y^2 = a^2$

$$x^2+y^2-4y+4 = 4$$

$$x^2+y^2-4y = 0$$

Substituting for x & y, we get

$$\Rightarrow \left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{v}{u^2+v^2}\right)^2 - 4\left(\frac{v}{u^2+v^2}\right) = 0$$

$$\Rightarrow \frac{u^2+v^2-4v(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$$\Rightarrow (u^2+v^2) - 4v(u^2+v^2) = 0$$

$$\Rightarrow (u^2+v^2)(1-4v) = 0$$

$$\Rightarrow 1-4v = 0$$

$$\Rightarrow v = \frac{1}{4}$$

which is a straight line in the w-plane.

(6) Find the image of the infinite strip $0 < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$

Sol: Given $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{1}{(u+iv)} \times \frac{u-iv}{(u-iv)}$$

$$x+iy = \frac{u}{u^2+v^2} - \frac{iv}{u^2+v^2}$$

Suppose $y=0 \Rightarrow v=0$

$$y=\frac{1}{2} \Rightarrow \frac{1}{2} = \frac{-v}{u^2+v^2}$$

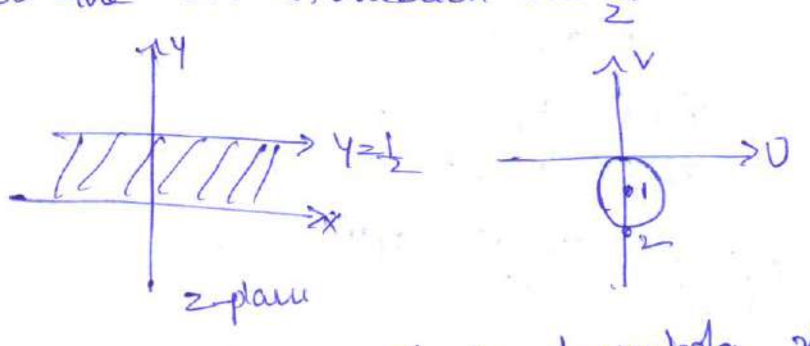
$$\Rightarrow u^2+v^2+2v=0$$

$$\Rightarrow u^2+(v+1)^2=1$$ which is circle with centre $(0,-1)$ and radius 1

Hence under transformation $w=\frac{1}{z}$

The straight line $y=0$ is transformed to line $v=0$ and the line $y=\frac{1}{2}$ is transformed to circle $u^2+(v+1)^2=1$

Hence the infinite strip $0<y<\frac{1}{2}$ in z-plane is mapped into the region b/w the line $v=0$ and the circle $u^2+(v+1)^2=1$ in w-plane under the transformation $w=\frac{1}{z}$



(7) Show that the image of the hyperbola $x^2-y^2=1$ under the transformation $w=\frac{1}{z}$ is the lemniscate $\rho^2=\cos 2\phi$

Soln: Let $z=re^{i\theta}$ and $w=\rho e^{i\phi}$

and $w=\frac{1}{z}$

$$\Rightarrow \rho e^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

$$\rho=\frac{1}{r}, \ \phi=-\theta$$

Given hyperbola is $x^2-y^2=1$ $\quad (\because x=r\cos\theta, \ y=r\sin\theta)$

$$\Rightarrow r^2\cos^2\theta - r^2\sin^2\theta=1$$

$$\Rightarrow r^2(\cos^2\theta-\sin^2\theta)=1$$

$$\Rightarrow r^2\cos 2\theta=1$$

$$\Rightarrow \frac{1}{\rho^2}\cos(-2\phi)=1$$

$$\Rightarrow \rho^2=\cos 2\phi$$

$\therefore$ The hyperbola $x^2-y^2=1$ in the z-plane is mapped into lemniscate $\rho^2=\cos 2\phi$ in the w-plane.

(8) Show that the transformation $w = z + \frac{1}{z}$ converts the straight line $\arg z = a$ $(|a| < \frac{\pi}{2})$ into a branch of the hyperbola of eccentricity $\sec a$. 5

Sol:- Given transformation is $w = z + \frac{1}{z} \rightarrow (1)$

w is analytic every where except at a simple pole $z = 0$

Since $\frac{dw}{dz} = 1 - \frac{1}{z^2}$

$w' = \frac{z^2 - 1}{z^2} = \frac{(z-1)(z+1)}{z^2}$

w' is non-zero every where except at $z = \pm 1$

Thus the mapping is conformal every where except at the point $z = \pm 1$

Let $z = re^{i\theta}$

$\Rightarrow w = z + \frac{1}{z}$

$= re^{i\theta} + \frac{1}{re^{i\theta}}$

$= re^{i\theta} + \frac{1}{r}e^{-i\theta}$

$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$

$u + iv = \cos\theta(r + \frac{1}{r}) + i\sin\theta(r - \frac{1}{r})$

Comparing real and imaginary parts, we get

$u = (r + \frac{1}{r})\cos\theta \rightarrow (2)$, $v = (r - \frac{1}{r})\sin\theta \rightarrow (3)$

To find the image, we have to eliminate r,

From (2) & (3), we have

$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = (r + \frac{1}{r})^2 - (r - \frac{1}{r})^2 = 4$

$r^2 + \frac{1}{r^2} + 2r\cdot\frac{1}{r} - r^2 - \frac{1}{r^2} + 2\cdot r\cdot\frac{1}{r} = 4$

Thus $\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1 \rightarrow (4)$

Eq (4) represents a hyperbola

For $\theta = \arg z = a$,

$(4) \Rightarrow \frac{u^2}{4\cos^2 a} - \frac{v^2}{4\sin^2 a} = 1 \rightarrow (5)$

From this we conclude that the radial lines $\theta = a$ $(< \frac{\pi}{2})$ in the z-plane are transformed into a family of confocal hyperbolas in the w-plane.

Here $a^2 = 4\cos^2 a$ & $b^2 = 4\sin^2 a$

Eccentricity e of the hyperbola is given by

$e^2 = 1 + \frac{b^2}{a^2}$

i.e $e^2 = 1$

i.e $e^2 = 1 + \frac{\sin^2 a}{\cos^2 a} = \frac{\cos^2 a + \sin^2 a}{\cos^2 a}$

$e^2 = \frac{1}{\cos^2 a}$

$e^2 = \sec^2 a$

$e = \sec a \cdot \; (a < \frac{\pi}{2})$

$\therefore$ Eccentricity of the hyperbola is $\sec a$.

Hence the given transformation maps the family of straight lines $\theta = a$ of the z-plane onto the family of confocal hyperbolas of the w-plane with eccentricity $\sec a$.

(9) Under the transformation $w = \frac{z-i}{1-iz}$. find the image of the circle (i) $|w|=1$ (ii) $|z|<1$ in the w-plane.

Sol:- (i) The given transformation is

$$w = \frac{z-i}{1-iz}$$

Unit circle in w-plane is $|w|=1$

i.e $\left|\frac{z-i}{1-iz}\right| = 1$

$|z-i| = |1-iz|$

$|(x+iy)-i| = |1-i(x+iy)|$

$x^2 + (y-1)^2 = (1+y)^2 + x^2$

$x^2+y^2-2y+1 = y^2+2y+1+x^2$

$-2y-2y = 0$

$-4y = 0$

$y = 0$ which is real axis in the z-plane.

(ii) Given $w = \frac{z-i}{1-iz}$

$w - iwz = z - i$

$w + i = z(1+iw) \Rightarrow z = \frac{w+i}{1+iw}$

when $|z| = 1$

$$\left|\frac{w+i}{1+iw}\right| = 1$$

$$|u+iv+i| = |1+i(u+iv)|$$

$$|u+i(v+1)| = |(1-v)+iu|$$

$$\cancel{u^2}+(v+1)^2 = (1-v)^2+\cancel{u^2}$$

$$v^2+2v+1 = 1-2v+v^2$$

$$4v = 0$$

$v = 0$ corres real axis of w-plane.

It corresponds to boundary of the circle $|z|=1$

(10) Show that the function $w=\frac{4}{z}$ transforms the straight line $x=c$ in the z-plane into a circle in the w-plane.

Sol: Given $w = \frac{4}{z}$

$$u+iv = \frac{4}{x+iy} = \frac{4}{(x+iy)}\times\frac{(x-iy)}{(x-iy)}$$

$$= \frac{4x}{x^2+y^2} - \frac{i4y}{x^2+y^2}$$

$$\therefore\ u = \frac{4x}{x^2+y^2} \rightarrow (1),\quad v = \frac{-4y}{x^2+y^2} \rightarrow (2)$$

when $x = c$, $\Rightarrow u = \frac{4c}{c^2+y^2} \Rightarrow y^2 = \frac{4c-c^2u}{u}$.

$$v = \frac{-4\left(\frac{4c-c^2u}{u}\right)}{c^2+\frac{4c-c^2u}{u}}$$

$$v = \frac{\frac{-16c+4c^2u}{u}}{\frac{c^2u+4c-c^2u}{u}}$$

$$v = \frac{-16c+4c^2u}{4c} = \frac{-4c(4-cu)}{4c}$$

$$(2) \Rightarrow v^2 = \frac{16y^2}{(x^2+y^2)^2}$$

$$v^2 = \frac{16(4c-c^2u)}{u}\times\frac{u^2}{16c^2}$$

$$v^2 = \frac{(4-cu)u}{c}$$

$$cv^2 = (4-cu)u$$

$$u^2+v^2-\frac{4}{c}u = 0$$

which is a circle in the w-plane.

Show that the transformation $w = \frac{2z+3}{z-4}$ changes the circle

(11) $x^2+y^2-4x=0$ into the straight line $4u+3=0$

Sol:- The given transformation is

$$w = \frac{2z+3}{z-4}$$

Solving for z,

$$wz-4w = 2z+3$$

$$\Rightarrow (w-2)z = 3+4w$$

$$\therefore \quad z = \frac{3+4w}{w-2} \longrightarrow (1)$$

Hence $$\bar{z} = \frac{3+4\bar{w}}{\bar{w}-2} \longrightarrow (2)$$

The eq of the circle $x^2+y^2-4x=0$ is rewritten as.

$$z\bar{z} - 2(z+\bar{z}) = 0 \longrightarrow (3)$$

Thus the image of the circle is

$$\frac{(3+4w)}{(w-2)} \cdot \frac{(3+4\bar{w})}{(\bar{w}-2)} - 2\left[\frac{3+4w}{w-2} + \frac{3+4\bar{w}}{\bar{w}-2}\right] = 0$$

i.e, $(3+4w)(3+4\bar{w}) - 2\left[(3+4w)(\bar{w}-2) + (3+4\bar{w})(w-2)\right] = 0$

i.e, $9+16w\bar{w}+12(w+\bar{w}) - 2\left[(-6+4w\bar{w}+3\bar{w}-8w)+4w\bar{w}+3w-6-8\bar{w}\right]=0$

$$22(w+\bar{w})+33=0$$

$$\Rightarrow 2(w+\bar{w})+3=0$$

$.\ 4u+3=0$ which is a straight line in the w-plane.

(12) Show that the relation $w = \frac{5-4z}{4z-2}$ transforms the circle $|z|=1$ into a circle of radius unity in the w-plane.

(13) Find and plot the rectangular region $0 \le x \le 1, 0 \le y \le 2$ under the transformation $w = \sqrt{2}\, e^{i\frac{\pi}{4}} z + (1-2i)$

Sol:- The given transformation is

$$w = \sqrt{2}\, e^{i\frac{\pi}{4}} z + (1-2i)$$

$$u+iv = \sqrt{2}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right)(x+iy) + (1-2i)$$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i1}{\sqrt{2}}\right)(x+iy) + (1-2i)$ 9

$= (1+i)(x+iy) + (1-2i)$

$u+iv = (x-y+1) + i(x+y-2)$

Here $\left.\begin{array}{l} u = x-y+1 \\ v = x+y-2 \end{array}\right\} \to (1)$

$0 \le x \le 1$

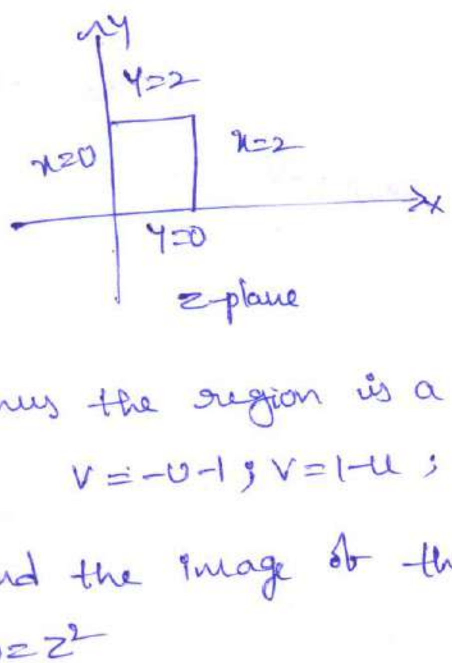
put $x=0$ in (1) $\left.\begin{array}{l} u = -y+1 \\ v = y-2 \end{array}\right\} \Rightarrow v = -u-1$

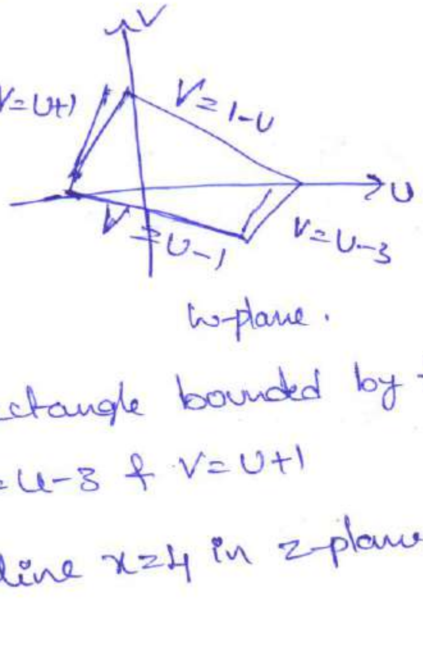
put $x=1$ in (1) $\left.\begin{array}{l} u = 2-y \\ v = y-1 \end{array}\right\} \Rightarrow v = 1-u$

$0 \le y \le 2$

put $y=0$ in (1) $\left.\begin{array}{l} u = x+1 \\ v = x-2 \end{array}\right\} \Rightarrow v = u-3$

put $y=2$ in (1) $\left.\begin{array}{l} u = x-1 \\ v = x \end{array}\right\} \Rightarrow v = u+1$





Thus the region is a rectangle bounded by the lines.

$v = -u-1$; $v = 1-u$; $v = u-3$ & $v = u+1$

(14) Find the image of the line $x=4$ in z-plane under the transformation $w=z^2$

The given transformation is

$w = z^2$

i.e $u+iv = (x+iy)^2$

$u+iv = (x^2-y^2) + i(2xy)$

comparing both sides real and imaginary parts

$u = x^2-y^2 \to (1)$
$v = 2xy \to (2)$

At the image of the line $x=4$

$① \Rightarrow \quad u = 16 - y^2 \rightarrow ③$

$② \Rightarrow \quad v = 8y \rightarrow ④$

from ③ & ④, eliminate y

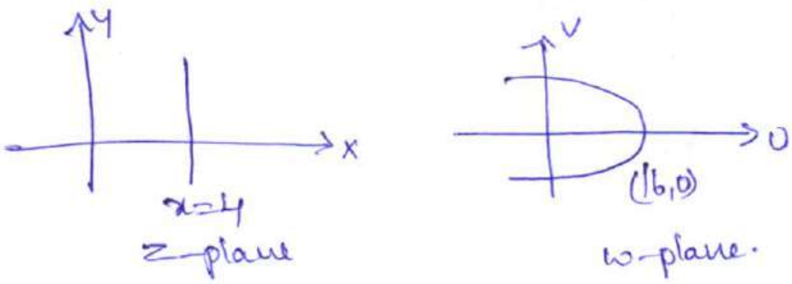
$$U = 16 - \left(\frac{V}{8}\right)^2$$

$$U = 16 - \frac{V^2}{64}$$

$$64U = 1024 - V^2$$

$$V^2 = -16(U - 4^2)$$

This represents a parabola in w-plane with vertex at (16,0) and focus at the origin and symmetrical about the real line.



(15) Find the image of the region in the z-plane between the lines $y=0$ and $y=\frac{\pi}{2}$ under the transformation $w=e^z$

Sol:- let $z = x+iy$ and $w = Re^{i\phi}$

Given transformation is

$$w = e^z$$

$$\therefore Re^{i\phi} = e^{x+iy} = e^x \cdot e^{iy}$$

Equating real and imaginary parts, we get

$$R = e^x \rightarrow (1)$$

$$\phi = y \rightarrow (2)$$

If $y=0$ then $\phi = 0$ represents radial line making an angle of zero radians with the x-axis.

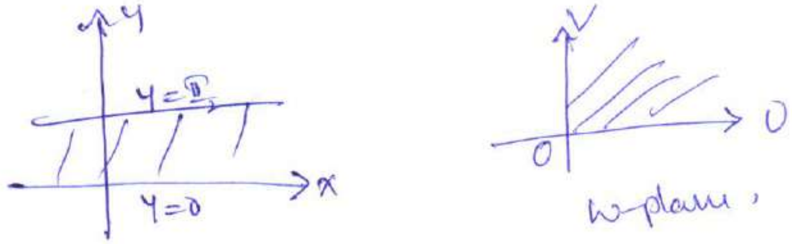
If $y = \frac{\pi}{2}$ then $\phi = \frac{\pi}{2}$ represents radial line making an angle of $\frac{\pi}{2}$ radians with the x-axis

Thus when $R = e^x$ increases from 0 to ∞ monotonically

as x takes values from $-\infty$ to ∞, 8

The line $y = \frac{\pi}{2}$ in z-plane is mapped onto the ray $\phi = \frac{\pi}{2}$

excluding the origin in w-plane.



z-plane

Hence the infinite strip bounded by the lines $y = 0$ & $y = \frac{\pi}{2}$

is mapped onto the upper quadrant of w-plane.

(16) The image of the infinite strip bounded by $x = 0$ & $x = \frac{\pi}{4}$

Under the transformation $w = \cos z$.

Sol:- Consider $w = \cos z$

$$u + iv = \cos(x + iy)$$

$$= \cos x \cos(iy) - \sin x \sin(iy)$$

$$u + iv = \cos x \cosh y - i \sin x \sinh y$$

Equating real and imaginary parts, we get

$$u = \cos x \cosh y \longrightarrow (1)$$

$$v = -\sin x \sinh y \longrightarrow (2)$$

when $x = 0$, $u = \cosh y$ & $v = 0$

when $-\infty < y < \infty \Rightarrow 1 \leq \cosh y < \infty$

$$\Rightarrow 1 \leq u < \infty$$

Thus the line $x = 0$ is transformed $v = 0$, $1 \leq u < \infty$

from (1) & (2) eliminating 'y', we have

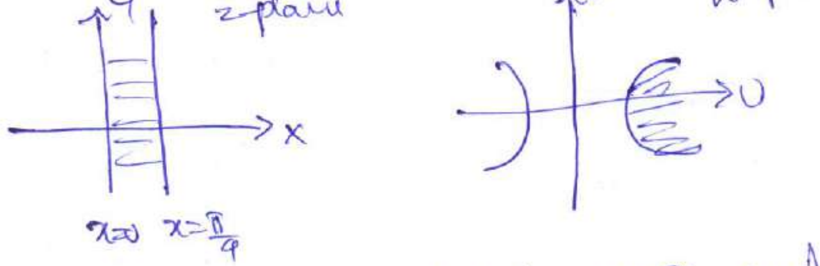
$$\frac{u^2}{\cos^2 x} - \frac{v^2}{\sin^2 x} = 1$$

when $x=\frac{\pi}{4}$, this becomes.

$$\frac{u^2}{\left(\frac{1}{2}\right)} - \frac{v^2}{\left(\frac{1}{2}\right)} = 1$$

$$\Rightarrow u^2 - v^2 = \frac{1}{2}$$

Thus the line $x=\frac{\pi}{4}$ is transformed to the rectangular hyperbola $u^2-v^2=\frac{1}{2}$ in the w-plane under the transformation $w=\cos z$



The infinite strip bounded by $x=0$ and $x=\frac{\pi}{4}$ in the z-plane is transformed to the region b/w the u-axis from 1 to ∞ and the rectangular hyperbola $u^2-v^2=\frac{1}{2}$

(17) Find and plot the image of the regions
(i) $x>1$, (ii) $y>0$, (iii) $0<y<(\frac{1}{2})$ under the transformation $w=\frac{1}{z}$

(or)

Find the image of the infinite strip, $0<y<(\frac{1}{2})$ under the mapping function $w=\frac{1}{z}$

Sol:- The given transformation is $w=\frac{1}{z}$

Let $w=u+iv$ & $z=x+iy$

$$z=\frac{1}{w}=\frac{1}{u+iv} \neq \frac{1}{u+iv}\times\frac{u-iv}{u-iv}$$

$$x+iy = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

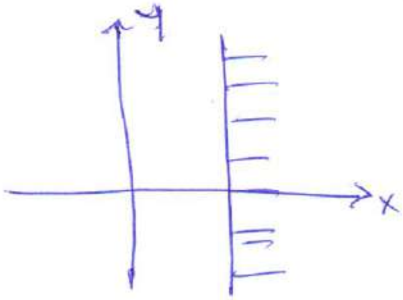
∴ Here $x=\frac{u}{u^2+v^2}$, $y=\frac{v}{u^2+v^2}$

(i) consider the region $x > 1$

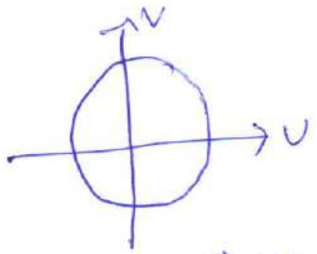
$\therefore \frac{u}{u^2+v^2} > 1 \Rightarrow u^2+v^2-u < 0$

But $u^2+v^2-u=0$ is a circle in the (u,v) plane with centre at $(\frac{1}{2},0)$ and radius $= \sqrt{\frac{1}{4}+1} = \frac{\sqrt{5}}{2}$

$\therefore u^2+v^2-u<0$ is the region in the (u,v) plane which is within the circle with centre of $(\frac{1}{2},0)$ and radius $\frac{\sqrt{5}}{2}$



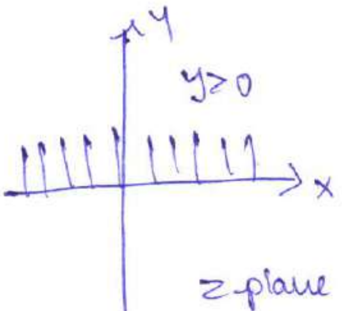
z-plane $x=1$

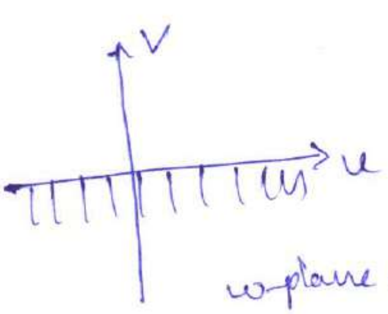


w-plane.

(ii) consider the region $y>0$

$y>0 \Rightarrow \frac{-v}{u^2+v^2} > 0 \Rightarrow -v>0 \Rightarrow v<0$





$\therefore$ The upper half of the z-plane is mapped onto the lower half of the w-plane.

(iii) consider the region $0<y<\frac{1}{2}$

$y>0 \Rightarrow v<0$

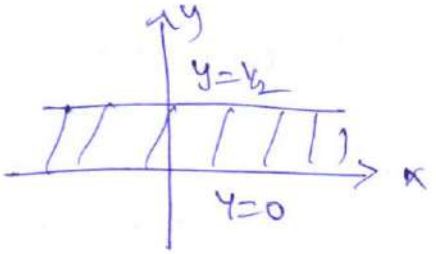
$y<\frac{1}{2} \Rightarrow \frac{-v}{u^2+v^2} < \frac{1}{2}$

$\Rightarrow u^2+v^2 > -2v$

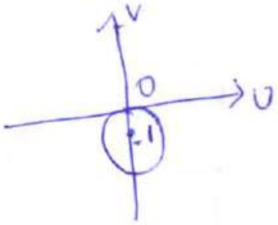
$\Rightarrow u^2+v^2+2v>0$

$u^2+v^2+2v=0$ is a circle in (u,v) plane with centre at $(0,-)$ and radius 1 unit

The region $0<y<\frac{1}{2}$ in the z-plane is the strip b/w the lines $y=0$ + $y=\frac{1}{2}$



The image of the above strip is the region with in the circle with the circle with centre at (0,-1) and radius 1 in the lower half of the w-plane.



Bilinear Transformation (or) Mobius Transformations –

The transformation $w = \frac{az+b}{cz+d}$ where a, b, c, d are complex constants and $ad-bc \neq 0$ is known as "bilinear transformation".

A bilinear transformation is conformal:–

consider the bilinear transformation $w = \frac{az+b}{cz+d}$

Diff w.r.to 'z', we have.

$$\Rightarrow w' = \frac{(cz+d)(a) - (az+b)c}{(cz+d)^2}$$

$$w' = \frac{ad-bc}{(cz+d)^2}$$

Since $ad-bc \neq 0$, we have $w' \neq 0$

Hence the mapping defined above is conformal.

If $ad-bc=0$ then $w'=0$ for all z and every point of z-plane will be critical.

special cases of Bilinear transformation:–

(i) $w = z+b$ (Translation)

(ii) $w = az+b$ (Linear transformation)

(iii) $w = az$ (Rotation)

(iv) $w = \frac{1}{z}$ (Inversion on a unit circle)

Invariant (or) Fixed point:–

For the mapping $w=f(z)$, the points that are mapped onto themselves are called "fixed points" (or) "invariant points".

i.e $w = f(z) = z$

Thus, all points such that $z = \frac{az+b}{cz+d}$ are called the invariant (or) fixed point of transformation (1).

i.e $cz^2 - (a-d)z - b = 0$

The quadratic eq of roots are called fixed points. If roots are equal, the bilinear transformation is said to be parabolic

A bilinear transformation preserves cross ratio property of four point

If t_1, t_2, t_3, t_4 are any four numbers then

$\Rightarrow \dfrac{(t_1-t_2)(t_3-t_4)}{(t_1-t_4)(t_3-t_2)}$ is said to be their cross ratio and is denoted by (t_1, t_2, t_3, t_4).

Let z_1, z_2, z_3, z_4 be points in the z-plane which are mapped onto the points w_1, w_2, w_3, w_4 respectively under the bilinear transformation

$$w = \frac{az+b}{cz+d}.$$

Linear fractional transformation:- Three given distinct point z_1, z_2, z_3 can always be mapped onto three prescribed distinct points w_1, w_2, w_3 by one and only one, linear fractional transformation $w = f(z)$. This mapping is given implicitly by the eqn

$$\boxed{\frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w_1} = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}}$$

(1) Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$. Hence find

(a) The image of $|z| < 1$

(b) concentric circles $|z| = r$ $(r > 1)$

(c) The invariant point of this transformation. (10)

Sol:- Given $z_1 = 1$, $z_2 = i$, $z_3 = -1$ & $z_4 = z$ (Say)

and $w_1 = i$, $w_2 = 0$, $w_3 = -i$ & $w_4 = w$ (Say)

we know that cross-ratio of four points is invariant under bilinear transformation

Thus, $$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}$$

Substituting the values, we get

$$(u^2+v^2)(1-r^2) - 2u(1+r^2) + (1-r^2) = 0$$

$$(u^2+v^2) - 2\left(\frac{1+r^2}{1-r^2}\right)u + 1 = 0$$

$x^2+y^2+2gx+2fy+c=0$
$u^2+v^2+2\left(\frac{1+r^2}{1-r^2}\right)u+2f.0+1=0$

Thus the image of $|z|=r$ is a circle in w-plane of radius.

$$= (g^2+f^2-c)^{1/2}$$

$$= \left[\left(\frac{1+r^2}{1-r^2}\right)^2 + 0 - 1\right]^{1/2} = \frac{2r}{1-r^2} \quad \text{and centre at } (-g,-f) = \left(\frac{1+r^2}{1-r^2}, 0\right)$$

(c) Put $w=z$ to get invariant points

$$\therefore \quad z = \frac{1+iz}{1-iz}$$

$$\Rightarrow iz^2 + (i-1)z + 1 = 0$$

$$\Rightarrow z = \frac{(1-i) \pm \sqrt{(i-1)^2 - 4i}}{2i} = -\frac{1}{2}\left[1+i \pm \sqrt{6i}\right]$$

$z = -\frac{1}{2}\left[1+i-\sqrt{6i}\right],\ -\frac{1}{2}\left[1+i+\sqrt{6i}\right]$ are called fixed points

(2) Find the bilinear transformation which transform the points $(\infty, i, 0)$ in the z-plane into the points $(0, i, \infty)$ in the w-plane.

Sol:- Let $(z_1, z_2, z_3) = (\infty, i, 0)$

and $(w_1, w_2, w_3) = (0, i, \infty)$

The required bilinear transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w-0)(i-\infty)}{(0-i)(\infty-w)} = \frac{(z-\infty)(i-0)}{(\infty-i)(0-z)}$$

$$\Rightarrow -\frac{w(i-\infty)}{i(\infty-w)} = \frac{(z-\infty).i}{(\infty-i)z}$$

$$-\frac{w}{i}\lim_{n\to\infty}\left(\frac{i-n}{n-w}\right) = \frac{-i}{z}\lim_{n\to\infty}\left(\frac{z-n}{n-i}\right) \quad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$-\frac{w}{i}\lim_{n\to\infty}\left(\frac{\frac{i}{n}-1}{1-\frac{w}{n}}\right) = \frac{-i}{z}\lim_{n\to\infty}\left(\frac{\frac{z}{n}-1}{1-\frac{i}{n}}\right)$$

$$-\frac{w}{i}\left(\frac{0-1}{1-0}\right) = \frac{-i}{z}\left(\frac{0-1}{1-0}\right) \Rightarrow$$

$$\frac{w}{i} = \frac{i}{z}$$

$$\therefore \boxed{w = -\frac{1}{z}}$$

$$\frac{(1-i)(-1-z)}{(1-z)(-1-i)} = \frac{(i-0)(-i-w)}{(i-w)(-i-0)}$$

$$\Rightarrow \frac{w+i}{w-i} = \frac{(z+1)(1-i)}{(z-1)(1+i)}$$

By componendo, & Dividendo, we get

$$\frac{2w}{2i} = \frac{(z+1)(1-i) + (z-1)(1+i)}{(z+1)(1-i) - (z-1)(1+i)} \quad \frac{Nr+dr}{Nr-dr}$$

$$\therefore w = \frac{1+iz}{1-iz}$$

which is required bilinear transformation.

(b)

$w = \frac{1+iz}{1-iz}$

$w - wiz = 1+iz$

$iz + wiz = w-1$

$z(i+wi) = w-1$

$z = \frac{w-1}{i(1+w)}$

$z = \frac{i(1-w)}{1+w}$

$\frac{a}{b} = \frac{c}{d}$

$\frac{a+b}{a-b} = \frac{c+d}{c-d}$

(a) z can be written as

$$z = i\left(\frac{1-w}{1+w}\right)$$

$$|z| = \left|\frac{i(1-w)}{(1+w)}\right| < 1$$

$$|i|\,|1-w| < |1+w| \qquad [\because |i| = 1]$$

$$|1-u-iv| < |1+u+iv|$$

$\sqrt{(1-u)^2+v^2} < \sqrt{(1+u)^2+v^2}$ take squaring on both side.

$$(1-u)^2+v^2 < (1+u)^2+v^2 \Rightarrow \cancel{1}-2u+\cancel{u^2}+\cancel{v^2} < \cancel{1}+2u+\cancel{u^2}+\cancel{v^2}$$

$$\Rightarrow \cancel{4u>0} \quad -4u<0 \Rightarrow u>0$$

Thus, the interior of the circle $x^2+y^2=1$ in the z-plane is mapped onto the entire half of the w-plane to the right of the imaginary axis.

(b) $|z| = r$ is transformed onto

$$\left|i\frac{(1-w)}{(1+w)}\right| = r$$

$$|i|\,|1-w| = r|1+w| \qquad |x+iy| = \sqrt{x^2+y^2}$$

$$|1-u-iv| = r\,|1+u+iv|$$

$$\sqrt{(1-u)^2+v^2} = r\sqrt{(1+u)^2+v^2}$$

Squaring on both si

$$\Rightarrow (1-u)^2+v^2 = r^2\left[(1+u)^2+v^2\right]$$

$$\Rightarrow (u^2+v^2)(1-r^2) + 1 - 2u = r^2(1+2u)$$

③ Find the Bilinear transformation which maps the points $(-1, 0, 1)$ into the points $(0, i, 3i)$ 12

④ Find the Bilinear transformation which maps the points $z_1 = 1,\ z_2 = i,\ z_3 = -1$ into the points $w_1 = 2,\ w_2 = i,\ w_3 = -2$

Sol: Let $w = \dfrac{az+b}{cz+d} \rightarrow ①$

Given $z_1 = 1, z_2 = i, z_3 = -1$ & $w_1 = 2, w_2 = i, w_3 = -2$

At $z_1 = 1, w_1 = 2$

$① \Rightarrow 2 = \dfrac{a+b}{c+d}$

$\Rightarrow 2c + 2d = a + b \rightarrow ②$

At $z_2 = i, w_2 = i$

$i = \dfrac{ai+b}{ci+d}$

$-c + di = ai + b \rightarrow ③$

At $z_3 = -1, w_3 = -2$

$-2 = \dfrac{-a+b}{-c+d}$

$2c - 2d = -a + b \rightarrow ④$

$② + ④ \quad 4c = 2b \Rightarrow b = 2c$

$② - ④ \quad 4d = 2a \Rightarrow a = 2d$

Substituting in ③,

$$-c + di = 2di + 2c$$

$$\cancel{di = 3c}\ -3c$$

$$d = \frac{-3c}{i}$$

Then $a = 2\left(\frac{-3c}{i}\right) = \frac{-6c}{i}$

Substituting the values a, b, d in ①, we obtain.

$$w = \frac{\frac{-6c}{i}z + 2c}{cz - \frac{3c}{i}} = \frac{-6cz + 2ic}{icz - 3c}$$

$\therefore \boxed{w = \dfrac{-6z + 2i}{iz - 3}}$

which is required bilinear transformation.

⑤ Determine the bilinear transformation that maps the points $(1-2i, 2+i, 2+3i)$ into the points $(2+i, 1+3i, 4)$

(6) Find the bilinear transformation which maps the points $(-i, 0, i)$ into the points $(-1, i, 1)$ respectively.

(7) Find the bilinear transformation which maps vertices $(1+i, 1-i, 2-i)$ of the triangle T of the z-plane into the point $(0, 1, i)$ of the w-plane.

(8) Find the bilinear transformation which transforms the points $z = 2, 1, 0$ into $w = 1, 0, i$ respectively.

(9) Determine the bilinear transformations whose fixed points are $1, -1$.

Sol: The equation in z with α, β as roots is

$$z^2 - (\alpha+\beta)z + \alpha\beta = 0.$$

For any complex constant γ

$$z^2 - (\alpha+\beta)z + \gamma z - \gamma z + \alpha\beta = 0$$

$$\Rightarrow z[z - (\alpha+\beta-\gamma)] = \gamma z - \alpha\beta.$$

$$\therefore \boxed{z = \frac{\gamma z - \alpha\beta}{z - (\alpha+\beta-\gamma)}} \longrightarrow (1)$$

For various values of γ, (1) gives bilinear transformation with fixed points α, β.

$\therefore$ The bilinear transformation with $\underset{\alpha}{1}, \underset{\beta}{-1}$ as fixed points as

$$w = \frac{\gamma z - (1)(-1)}{z - (1-1-\gamma)}$$

$$\boxed{w = \frac{\gamma z + 1}{z + \gamma}}$$ for various values of γ

i.e for $\gamma = 0 \Rightarrow w = \frac{1}{z}$

$\gamma = 1 \Rightarrow w = \frac{z+1}{z+1}$

& $\gamma = 2 \Rightarrow w = \frac{2z+1}{z+2}$

(10) Find the bilinear transformation whose fixed points are $1, i$ and maps 0 to -1

Sol:- The bilinear transformation with α, β as fixed points given by

$$w = \frac{\gamma z - \alpha\beta}{z-(\alpha+\beta-\gamma)} \quad \text{for various values of } \gamma$$

For $\alpha=1, \beta=i$, we have

$$w = \frac{\gamma z - i}{z-1-i+\gamma}$$

Since $z=0$ is mapped to $w=-1$, we have.

$$-1 = \frac{0-i}{0-1-i+\gamma} \Rightarrow 1+i-\gamma = -i \Rightarrow -\gamma = -2i-1$$
$$\Rightarrow \gamma = 2i+1$$

Hence the required bilinear transformation is

$$w = \frac{(2i+1)z - i}{z-1-i+(2i+1)} = \frac{(2i+1)z-i}{z+i}$$

(11) Find the fixed points (or Invariant points) of the transformation.

(i) $w=\frac{2i-6z}{iz-3}$ (ii) $w=\frac{6z-9}{z}$ (iii) $w=\frac{z-1}{z+1}$ (iv) $w=\frac{2z-5}{z+4}$

Sol:- The fixed points of the transformation are such that the image of z is z itself

(i) put $w=z$ in the given transformation, we get

$$w = \frac{2i-6z}{iz-3}$$
$$\Rightarrow z = \frac{2i-6z}{iz-3}$$
$$\Rightarrow iz^2-3z = 2i-6z$$
$$\Rightarrow iz^2+3z-2i=0$$
$$az^2+bz+c=0$$

$$\therefore z = \frac{-3 \pm \sqrt{9-4(i)(-2i)}}{2i}$$
$$z = \frac{-3\pm\sqrt{9-8}}{2i}$$
$$z = \frac{-3-1}{i} = \frac{-2}{i} = 2i, \quad z = \frac{-3+1}{i} = \frac{-1}{i} = i$$

$\therefore$ Hence fixed points are $i, 2i$